In principle (and approximately), the visibility measured with a radio interferometer is the two-dimensional Fourier transform of the brightness distribution:

\[ V(u, v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I(x, y) e^{-i2\pi(ux+vy)} \, du \, dv \]

Consequently, the brightness distribution should be possible to obtain by the inverse Fourier transformation:

\[ I(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} V(u, v) e^{i2\pi(ux+vy)} \, du \, dv \]
Fourier transform pairs (1)

\( l(x, y) \) | \( |V(u, v)| \)

\( \delta \) function | uniform distribution

Gaussian | Gaussian

(figures: Amy Miodyszewski & David Wilner)
Fourier transform pairs (2)

\[ I(x, y) \quad |V(u, v)| \]

\[
\begin{align*}
\text{Gaussian ellipse} & \quad \text{Gaussian ellipse} \\
\text{disk} & \quad \text{Bessel function}
\end{align*}
\]

(kuvat: Amy Miodyszewski& David Wilner)
Aperture synthesis in practice (1)

- The visibility, \( V(u, v) \), is a complex function: \( V = |V|e^{i\Phi} \)
- Because the product \( P_n I_\nu \) (beam pattern \( \times \) brightness) is real, \( V(-u, -v) = V^*(u, v) \)
  \( (V \) is hermitean - the order of antenna does not matter)\n- \( V(u, v) \) is only known in a limited region and for discrete set of points in the \((u, v)\) plane.
- Therefore, we cannot calculate the brightness distribution by the inverse Fourier transform.
Sampling function and dirty image

- By defining a sampling function \( g(u, v) \) so that \( g \neq 0 \) only in points \((u_k, v_k)\) where \( V \) has been measured, the product \( gV \) is defined in the whole \((u, v)\) plane.
- The inverse Fourier transform of the product \( gV \) can be calculated:

\[
I_D = \int_{-\infty}^{+\infty} \int g(u, v) V(u, v) e^{i2\pi(ux+vy)} dudv
\]
- The resulting function \( I_D(x, y) \) is called dirty image
According to the convolution theorem of the Fourier transform, the dirty image is the convolution of the true image (of the brightness distribution), $I(x, y)$, and the synthetic beam, $P_{\text{syn}}$:

$$I_D = I(x, y) \otimes P_{\text{syn}},$$

where

$$P_{\text{syn}} \equiv \int_{-\infty}^{+\infty} g(u, v) e^{i2\pi(ux+vy)} \, dudv.$$
Synthetic beam (2)

- The sampling function, $g(u, v)$, can be written in terms of the $\delta$ function:

$$g(u, v) = \frac{\sum_{k=0}^{N-1} w_k \delta(u - u_k, v - v_k)}{\sum_{k=0}^{N-1} w_k}$$

where $w_k$ are weight coefficients.

- The normalization $\int \int g(u, v) \, du \, dv = 1$ implies that $P_{\text{syn}}(0, 0) = 1$ (as for the normalized beam pattern of a single-dish telescope).

- The half-power width of the synthetic beam ("main" beam) is $\Delta x \sim 1/u_{\text{max}}, \Delta y \sim 1/v_{\text{max}}$

- The "side lobes" extend across the whole sky.
Example 1

\[ I(x, y) \rightarrow \rightarrow \rightarrow I_D(x, y) \]

(figures: Amy Miodyszewski & David Wilner)
Incomplete $uv$ coverage:

- “Central hole”: the total flux of an extended flux is not recovered because of the missing zero spacings:
  \[ V(0, 0) = \int_{\text{source}} I(x, y) \, dx \, dy \]
- The largest scales that can be imaged: $\Delta \alpha < 1/u_{\text{min}}$, $\Delta \delta < 1/v_{\text{min}}$
- Angular resolution: $\Delta \alpha \sim 1/u_{\text{max}}$, $\Delta \delta \sim 1/v_{\text{max}}$
- The gaps in the $uv$ plane create side lobes.
Example 2

Brightness distribution: two point sources at the offsets \((\Delta \alpha, \Delta \delta) = (0'', 0'')\) and \((10'', 10'')\), \(\delta = 80^\circ\) from the phase centre
Measurement \(H = -2^h \rightarrow +2^h\)
Antenna positions: 0, 90, 170, 210, 225 m in the East-West direction
Baselines: \(B=15, 40, 55, 80, 90, 120, 135, 170, 210\) and 225 m
Wavelength: \(\lambda = 1\) cm

Dirty image  |  Synthetic beam  |  Cleaned image
Example 3

Brightness distribution: Gaussian surface density, FWHM (full width at half-maximum) 10′′, \((\Delta \alpha, \Delta \delta) = (0′′, 0′′)\), emits spectral line emission, total flux density 1 Jy at the line peak.

Measurement near the meridian \(H \sim 0^h\)

Antennas: 0, 90, 150, 400 m, East-West

Wavelength: \(\lambda = 1.3\text{cm}\)

Baselines: B=90, 150, 60, 400, 310, 250 m

\(\lambda/B \approx 29″, 17″, 43″, 7″, 8″\) and 10″

Spectra with different baselines
Visibility weighting (1)

- Sampling function $g(u, v)$:

$$g(u, v) = \frac{\sum_{k=0}^{N-1} w_k \delta(u - u_k, v - v_k)}{\sum_{k=0}^{N-1} w_k}$$

- The weight coefficients $w_k$ affect the synthetic beam and the side lobes.

- **Natural weighting**: $w_k = 1/\sigma_k^2$
  ($\sigma$ is the RMS noise)

- **Uniform weighting**: $w_k = 1/\rho(u_k, v_k)$
  ($\rho$ is the density of measurement points in the neighbourhood of $(u, v)$)
Visibility weighting (2)

- In addition, long baselines can be given smaller weights in order to create a larger synthetic beam. This is called tapering or the \textit{uv} plane.

- The weight function $w_k$ can be then written as

$$w_k = T_k D_k ,$$

$T_k$ is the \textit{tapering function} of the \textit{uv} plane.  
$D_k$ is \textit{density weighting function} of the measured points.

- The angular resolution can be “worsened” using the function $T_k$, for example to help comparison with some previous data, or to achieve a better signal-to-noise ratio.
Visibility weighting (2)

- The most common tapering function is Gaussian:
  \[ T(r) = \exp(-r^2/2\sigma^2) \]
- In CASA (and Miriad) the desired angular resolution in the image plane is given (FWHM of the synthesized beam)
- In AIPS the tapering function is given in the \( uv \) plane
  Relationship: \( \text{FWHM [radians]} = 0.37/\sigma \text{[wavelengths]} \) or \( \text{FWHM ["]} = 0.77\lambda \text{[cm]}/\sigma \text{[km]} \)
Density weighting function $D_k$ (1)

- **Natural weighting** is good for detecting weak compact sources (the best S/N for a point source)
- **Uniform weighting** gives a better image quality if the source has both large scale and small scale structure
- A hybrid scheme, **robust weighting**, is a compromise of the two trying to attain a narrow main beam and weak side lobes.
Density weighting function $D_k$ (2)

ATCA 1933-400, contours 5% and 50%
Tapering function $T_k$

ATCA 1933-400, contours 5% and 50%

natural FWHM 10"

uniform FWHM 10"
In aperture synthesis the antennas track a certain point in the sky.

Because of the earth rotation complex source visibilities are measured with a large number of projected baselines.

A science target and a calibration source alternate as the phase centre.

The calibrator is typically a strong point source, a planet, or its satellite.
Interferometric mapping (2)

- **Flux density calibrator** (primary calibrator): stable, unpolarized source with well known flux density

- **Phase calibrator** (secondary calibrator): a strong point source lying close to the science target. Used to correct the gain and phase variations.

- **Bandpass calibrator**: Very strong point source used to determine the response of the receiver as a function of frequency (bandpass function)
The amplitude and phase variations in each receiver (in the antennas), owing to instrumental effects and the atmosphere, are described by a complex gain $G$:

$$G(t, \nu) = A(t)B(\nu)e^{i2\pi \tau(t)(\nu-\nu_0)}$$

- $A(t)$ is the complex “antenna gain”
- $B(\nu)$ is the bandpass function
- $\tau(t)$ delay term
- $\nu_0$ central frequency of the band
Calibration (2)

- In addition, a part of $X(L)$ polarized radiation leaks into the $Y(R)$ receiver, and vice versa (instrumental polarization).
- The terms $A(t)$ and $\tau(t)$ exhibit rapid changes, $B(\nu)$ and instrumental polarization change slowly.
- In calibration $G(t, \nu)$ is solved using point sources.
- The solutions are interpolated to the science target.
Calibration example (1)

Flux density calibrator 1934-634 - original visibility data

-amplitude

-phase

-flag the beginning
Calibration example (2)

Flux density calibrator 1934-634 - original and corrected data

- For corrected data the phases of the XX and YY polarization products are close to zero
Calibration example (3)

Flux density calibrator 1934-634 - original and corrected data

- For corrected point source data the real parts of XX and YY polarization products are close to the true flux density (note the scale on the y axis at right)