Introduction to Particle Physics I
relativistic kinematics

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outline

• Lecture I: Orientation, Units, Elementary Interactions
• Lecture II: Relativistic kinematics
• Lectures III: Lorentz invariant scattering cross section
• Lecture IV: Accelerators and collider experiments
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Lecture II; Relativistic kinematics

- Particle decay
- Two-particle scattering
  - Scattering angle
  - Elastic scattering
  - Angular distribution
  - Relative velocity
  - Center of mass and laboratory systems
- Crossing symmetry
  - Interpretation of antiparticle-states
relativistic kinematics

references:
• Nachtmann [I.4], Hagedorn [II.1], Byckling & Kajantie [II.2]

notations:
\[ x^\mu = (x^0 = t, x^1, x^2, x^3) = (t, \vec{x}) \] contravariant four-vector
\[ x_\mu = (t, -\vec{x}) \] covariant four-vector
\[
g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \] metric tensor
\[
\tau^2 = t^2 - \vec{x}^2 = g_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu = x^2 \] Lorentz invariant
\[
d\tau = dt \sqrt{1 - \left(\frac{d\vec{x}}{dt}\right)^2} = \frac{dt}{\gamma} \] proper time
notations

The four-velocity: \( u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} \frac{dt}{d\tau} = \gamma(1, \tilde{v}) \)

Since  \( u^2 = \gamma^2 (1 - \tilde{v}^2) = 1 > 0 \), \( u \) is a time-like vector.

The four-momentum is defined as:

\[
p^\mu = mu^\mu = m\gamma(1, \tilde{v}) = (p^0 = E, \tilde{p})
\]

By calculating the corresponding Lorentz invariant,

\[
p^2 = m^2 u^2 = m^2 = E^2 - \tilde{p}^2,
\]

we find the energy-momentum relation

\[
E = \sqrt{m^2 + \tilde{p}^2}
\]

A particle is said to be relativistic if \( \tilde{p}^2 >> m^2 \). For a non-relativistic particle \( \tilde{p}^2 << m^2 \), and

\[
E = \sqrt{m^2 + \tilde{p}^2} = m \left( 1 + \frac{1}{2} \frac{\tilde{p}^2}{m^2} + \ldots \right) = m + \frac{1}{2} \frac{\tilde{p}^2}{m} + \ldots
\]

i.e. we recover the expression for \( |\tilde{v}| << 1 \) of Newtonian mechanics.
energy-momentum

- energy-momentum four vector

\[ p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p}) \]

- where \( E \) is the energy of the particle and \( p_x, p_y \) and \( p_z \) are the components of particle momentum.

\[ p^\mu \cdot p_\mu = E^2 - p_x^2 - p_y^2 - p_z^2 = m^2 \]

- length of a four-vector is an *invariant* – it does not change under Lorentz transformation
Particle decay

The four-momentum of a decaying particle - in its rest frame - is given by \( p = (M, 0, 0, 0) \).

Experimentally: \( \tau_{\pi^+ \rightarrow \mu^+ \mu^-} = 2.6 \times 10^{-8} \text{s} \)

\( E_\pi = 20 \text{GeV}, \ \gamma = \frac{E_\pi}{m_\pi} = 143 \Leftrightarrow v = 0.9999 \)

\[ \Rightarrow \frac{t_{\pi'}}{t_\pi} = 143. \]

The decay time - lifetime - is:

\[ d\tau^2 = dt^2 (1 - \bar{v}^2) \]

where \( dt^2 \) is the lifetime in laboratory frame:

\[ dt = \gamma d\tau > d\tau. \]
Constraints

Constraints:

(i) energy-momentum conservation \( p = p_1 + p_2 \)

and

(ii) mass-shell condition \( p_i^2 = m_i^2 \)

\[
p^2 = M^2 \quad p_1^2 = m_1^2 \quad p_2^2 = m_2^2 \\
p = (M, \bar{0}) \quad p_1 = (E_1, \bar{p}_1) \quad p_2 = (E_2, \bar{p}_2)
\]
constraints...

Therefore:

\[ p \cdot p_i = ME_i \Rightarrow E_i = \frac{1}{M} p \cdot p_i = \frac{1}{M} (p_1 \cdot p_i + p_2 \cdot p_i) \]

By using: \( p_1 \cdot p_2 = \frac{1}{2} \left( (p_1 + p_2)^2 - p_1^2 - p_2^2 \right) = \frac{1}{2} \left[ M^2 - m_1^2 - m_2^2 \right] \), we get

\[ E_1 = \frac{1}{2M} \left( p_1^2 + p_1 \cdot p_2 \right) = \frac{1}{2M} \left( M^2 + m_1^2 - m_2^2 \right) \]

\[ E_2 = \frac{1}{2M} (M^2 - m_1^2 + m_2^2) \]

we get: \( \tilde{p_1}^2 = E_1^2 - m_1^2 = \frac{1}{4M^2} \left( M^4 - 2M^2 (m_1^2 + m_2^2) + (m_1^2 - m_2^2) \right) = \tilde{p_2}^2 \)

i.e. only the directions of \( \tilde{p}_1 \) and \( \tilde{p}_2 \) remain unknown, while the energies and the absolute values of the momenta can be directly calculated.
two particle scattering

\[ p_i^2 = m_i^2 \quad (i = 1, \ldots, 4) \]

\[ p_1 + p_2 = p_3 + p_4 \]

For elastic scattering \( m_1 = m_3 \) and \( m_2 = m_4 \). Next consider the Lorentz invariants:

\[ p_i^2 = m_i^2 \quad \text{and} \quad \frac{p_1 \cdot p_2, p_1 \cdot p_3, p_1 \cdot p_4, p_2 \cdot p_3, p_2 \cdot p_4, p_3 \cdot p_4}{6 \text{ invariants, 2 linearly independent, 4 linearly dependent}} \]

The Mandelstam variables

\[ s = (p_1 + p_2)^2 \]
\[ t = (p_1 - p_3)^2 \]
\[ u = (p_1 - p_4)^2 \]
\[ s + t + u = \sum_{i=1}^{4} m_i^2 \]

The center of mass (c.m.s.) frame is defined by:

\[ \vec{p}_1^2 + \vec{p}_2^2 = 0 = \vec{p}_3^2 + \vec{p}_4^2 \]
Mandelstam variables: $s$, $t$, $u$

- two body scattering. $A + B \rightarrow C + D$

\[ p_A + p_B \rightarrow p_C + p_D \]

- scalar products of 4-vectors are invariants
- possible combinations:

\[ p_A \cdot p_B \quad p_A \cdot p_C \quad p_A \cdot p_D \]

- total 4-momentum is conserved $\Rightarrow$ there are only two independent Lorentz-invariant kinematic variables on which the reaction cross section can depend
Mandelstam variables: $s$, $t$, $u$

Three convenient variables:

$$s = (p_A + p_B)^2$$
$$t = (p_A - p_C)^2$$
$$u = (p_A - p_D)^2$$

for which:

$$s + t + u = M_A^2 + M_B^2 + M_C^2 + M_D^2$$

The Mandelstam variables nicely relate to the propagator masses in the leading order diagrams.
cross sections and luminosity

Cross section $\sigma$ can be defined by:

$$\text{number of events} = \sigma \cdot \mathcal{L}$$

or equivalently

$$\text{number of events per unit time} = \sigma \cdot \frac{d\mathcal{L}}{dt}$$

where an ”event” is an interaction such as pp scattering, $\mathcal{L}$ is the luminosity, i.e. ”number of chances of an event per unit area”. For a fixed target within the beam of incident particles $\frac{d\mathcal{L}}{dt} = NJ$, where $N$ is the number of target particles and $J$ is the flux per unit area of particles in the incident beam.
cross sections and luminosity
frames of reference

The center of mass frame is defined by:

\[ \tilde{p}_1^2 + \tilde{p}_2^2 = 0 = \tilde{p}_3^2 + \tilde{p}_4^2 \]

In the c.m.s. frame, the variables are often denoted by an asterix: \( p_i^{\text{cms}} = p_i^* \)

In the laboratory frame, \( \tilde{p}_2 = 0 \) ("fixed target"), the variables are labelled with an L: \( p_i^{\text{lab}} = p_i^L \)

In deep inelastic processes (DIS), the Breit system, \( \tilde{p}_1 + \tilde{p}_3 = 0 \), is used and particle momenta labelled as \( p_i = p_i^B \)

In the c.m.s. frame: \( \tilde{p}_1^* = -\tilde{p}_2^* = \tilde{p}, \tilde{p}_3^* = -\tilde{p}_4^* = \tilde{p}' \) and

\[
\begin{align*}
p_1 &= (E_1^*, \sqrt{\tilde{p}^2 + m_1^2}, \tilde{p}) \\
p_2 &= (E_1^*, \sqrt{\tilde{p}^2 + m_2^2}, -\tilde{p}) \\
p_3 &= (E_3^*, \tilde{p}') \\
p_4 &= (E_4^*, -\tilde{p}')
\end{align*}
\]
two particle scattering

\[ p_1 + p_2 = \frac{(E_1^* + E_2^*, 0)}{\sqrt{s}} \text{ is no Lorentz invariant, whereas } s = (p_1 + p_2)^2 = (E_1^* + E_2^*)^2 \text{ is one.} \]

We can now express \( E_i^*, |\vec{p}|, \text{ and } |\vec{p}'| \) in terms of \( s \) (see exercises no.1.1):

\[
E_{1,3}^* = \frac{1}{2\sqrt{s}} (s + m_{1,3}^2 - m_{2,4}^2),\quad \vec{p}^2 = (E_1^*)^2 - m_1^2 = \frac{1}{4s} \lambda(s, m_1^2, m_2^2), \text{ where we use the Källén (triangle) function:}
\]

\[
\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ac = [a - (\sqrt{b} + \sqrt{c})^2][a - (\sqrt{b} - \sqrt{c})^2] = a^2 - 2a(b + c) + (b - c)^2
\]
two particle scattering

The Källén function has the following properties:

* symmetric under \( a \leftrightarrow b \leftrightarrow c \) and
* asymptotic behaviour: \( a \gg b, c: \lambda(a, b, c) \to a^2 \)

This allows some properties of the scattering process to be determined. From \( \tilde{p}^2, \tilde{p}'^2 > 0 \) it follows:

\[
    s_{\text{min}} = \max \left\{ (m_1 + m_2)^2, (m_3 + m_4)^2 \right\} \geq 0
\]

is the threshold of the process in the s-channel. At the high energy limit, \( s \gg m_i^2 \), one obtains:

\[
    E_1^* = E_2^* = E_3^* = E_4^* = |\tilde{p}| = |\tilde{p}'| = \frac{\sqrt{s}}{2}
\]
scattering angle

In the c.m.s. frame, the scattering angle $\Theta^*$ is defined by

$$\vec{p} \cdot \vec{p}' = |\vec{p}| |\vec{p}'| \cos \Theta^*$$

By using

$$p_1 \cdot p_3 = E_1^* E_3^* - |\vec{p}_1^*| |\vec{p}_3^*| \cos \Theta^*$$

$$t = (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2 p_1 \cdot p_3 = (p_2 - p_4)^2$$

we derive $\cos \Theta^* = function(s, t, m_i^2)$

$$\cos \Theta^* = \frac{s(t - u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda(s, m_1^2, m_2^2)} \sqrt{\lambda(s, m_3^2, m_4^2)}}$$

On the basis of the above, $2 \rightarrow 2$ scattering is described by two independent variables:

$$\sqrt{s} \text{ and } \Theta^* \quad \text{or} \quad \sqrt{s} \text{ and } t$$
elastic scattering

In elastic scattering, \( m_1 = m_3 \) and \( m_2 = m_4 \) (e.g. \( ep \rightarrow ep \)), and

\[
E_1^* = E_3^*, \quad E_2^* = E_4^*
\]

\[
|\vec{p}|^2 = |\vec{p}'|^2 = \frac{1}{4s} \left( s - (m_1 + m_2)^2 \right) \left( s - (m_1 - m_2)^2 \right)
\]

giving for the scattering angle in elastic scattering:

\[
t = (p_1 - p_3)^2 = -(\vec{p}_1 - \vec{p}_3)^2 = -2\vec{p}^2(1 - \cos \Theta^*)
\]

\[
\Rightarrow \quad \cos \Theta^* = 1 + \frac{t}{2|\vec{p}|^2}
\]

Relation to the physically allowed region yields:

\[
\begin{align*}
-1 &\leq \cos \Theta^* \leq 1 \\
\vec{p}^2 &\geq 0
\end{align*}
\]

\[
\begin{align*}
-4|\vec{p}|^2 &\leq t \leq 0 \\
s &\geq (m^1 + m^2)^2
\end{align*}
\]
The angular distribution is rotationally invariant with respect to the axis defined by the $\vec{p}$ - vector, i.e. $\int d\phi = 2\pi$

$$d\Omega^* = 2\pi d\cos\Theta^*$$

$$\frac{d\Omega^*}{dt} = \frac{4\pi s}{\sqrt{\lambda(s,m_1^2,m_2^2)} \sqrt{\lambda(s,m_3^2,m_4^2)}} = \frac{\pi}{|\vec{p}| \cdot |\vec{p}'|}$$
relative velocity

The relative velocity will be of relevance in defining the particle flux,

\[ v_{12} = |\vec{v}_1 - \vec{v}_2| = \left| \frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right| = \left| \frac{\vec{p}_1^*}{E_1^*} - \frac{\vec{p}_2^*}{E_2^*} \right| = \frac{|\vec{p}_1^*|}{E_1^* E_2^*} \left( \frac{E_1^* + E_2^*}{\sqrt{s}} \right) \]

from which we get,

\[ v_{12} E_1^* E_2^* = \sqrt{s} |\vec{p}_1^*| = \sqrt{s} \sqrt{E_1^*^2 - m_1^2} = \sqrt{s} \sqrt{\frac{1}{4s} (s + m_1^2 - m_2^2)^2 - m_1^2} \]

\[ = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \quad \text{The Moller flux factor.} \]

Note: The Moller flux factor is needed for normalizing the cross sections, since the classical volume element is not Lorentz invariant.
CMS and LAB systems

For the c.m.s. and laboratory systems:

\[ s = (E_1^* + E_2^*)^2 = (\text{total energy})^2 \]

\[ s_{\text{lab}} = m_1^2 + m_2^2 + 2m_2 E_1^L \]

\[ \overset{E_1^L \gg m_1, m_2}{\Rightarrow} 2m_2 E_1^L \]

An example: Fixed target and colliding beams mode at the Fermilab Tevatron, \( E_{\text{beam}} = 980 \text{ GeV} \).
CMS and LAB systems

fixed target: $m_2 = m_p$
- secondary beams!

\[ \sqrt{s_{p\bar{p}}} \bigg|_{\text{collider}} = 1960 GeV > 2 \times m_W \]

\[ \sqrt{s_{pN}} \bigg|_{\text{fixed target}} = 42.7 GeV < m_W \]

s-channel
crossing symmetry

t-channel

the 2→2 scattering process exhibits underlying symmetries
crossing symmetry

Example: When we exchange $p_3$ and $p_4$, s is not affected but t and u interchange their roles. Examine s-channel reaction (previous page): $1 + 2 \rightarrow 3 + 4$, for which the 4-momentum is conserved:

$$p_1 + p_2 = p_3 + p_4$$

The only positive Mandelstam variable for this reaction is s, hence the notation s-channel. $T_s$ describes the scattering dynamics of the process and will be discussed more later. It depends on the three Mandelstam variables and is predicted theoretically (QCD, QCD, EW, SUSY,...),

$$T_s(s,t,u) = T(s,t,u)_{s>0,t\leq0,u\leq0}$$

$T$ can then be extended analytically to the whole range $s,t,u \in \mathbb{R}$. Depending on the region, it can then describe different crossed reactions.

For instance, suppose we exchange $p_2$ and $p_3$, we then get naively,

$$p_1 + (-p_3) = (-p_2) + p_4$$
crossing symmetry

We now make the interpretation:

\[-p_n = p_{\bar{n}}\]

in which \(\bar{n}\) stands for the antiparticle of the particle \(n\), leading to the expression:

\[p_1 + p_3 = p_2 + p_4\]

Since 1 and 3 are the incoming particles, we speak of the "t-channel" process. We have:

\[T_t(s,t,u) = T(s,t,u)_{s\leq 0, t > 0, u \leq 0}\]
anti-particle states

The particles with 4-momentum $-p$ are interpreted as antiparticles with 4-momentum $p$. The reason for that becomes clear when we look at the 4-current,

$$ j^\mu = \left( \frac{\rho}{j} \right)_{QM} = \begin{cases} -e & \text{electron charge} \\ i(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*) & \text{probability density} \\ \rho & \text{charge density} \end{cases} $$

Inserting the wave function of the free electron, $\varphi = Ne^{-ip^r x}$, in the definition of the 4-current, we get

\begin{align*}
e^- \text{ with 4-momentum } + p^\mu : j^\mu(e^-) &= -2e|N|^2 p^\mu = -2e|N|^2 \left( + \frac{E}{+ \bar{p}} \right) \\
e^+ \text{ with 4-momentum } + p^\mu : j^\mu(e^+) &= +2e|N|^2 p^\mu = +2e|N|^2 (-p)^\mu \\
e^- \text{ with 4-momentum } -p^\mu : j^\mu(e^-) &= -2e|N|^2 (-p)^\mu = -2e|N|^2 \left( - \frac{E}{- \bar{p}} \right)
\end{align*}

And hence the rule:

$$ j^\mu(e^+) = j^\mu(e^-) \text{ with the substitution } p^\mu \rightarrow -p^\mu $$

Note: The whole 4-vector $p^\mu$ takes a minus sign, not only the spatial part.
- A particle with 4-momentum $-p^\mu$ is a representation for the corresponding antiparticle with 4-momentum $p^\mu$.
- Alternatively: Emission of a positron with energy $+E$ corresponds to the absorption of an electron with energy $-E$ (figure above).
In the Dalitz plot, the three reactions (s-, t- and u-channel ones) are described by a single diagrammatic representation. Function $T(s,t,u)$, evaluated in the relevant kinematical region, describes all three.
example: Moller & Bhabha scattering

Moller: $e^-e^-\rightarrow e^-e^-$ -crossing symmetry- Bhabha: $e^+e^-\rightarrow e^+e^-$
**NEXT:** Lecture III; Lorentz invariant scattering cross section and phase space

- **S-operator**
- Fermi’s golden rule
  - Total decay rate
  - Scattering cross section
  - Invariant phase space for $n_f$–particles
  - Differential cross section
- $2 \rightarrow 2$ scattering cross section
  - Phase space
  - Differential cross section
- Unitarity of the **S**-operator