Nuclear Structure and Nuclear Models

Introduction

Liquid drop model
Fermi-gas model
The Nuclear Shell model; experimental evidence

Nuclear Collective Motion, Criteria

Multipole Modes

The Nuclear Shell Model

Single-particle SM
Central potential: Square well
The harmonic oscillator potential
Summary of sp-potentials
The spin-orbit coupling: describing real nuclei
Woods-Saxon potential
Pairing Interaction
Nuclear Structure and Nuclear Models
Introduction
The nucleus as a many-body system from a number of viewpoints:

- Collective motion of nucleons $\rightarrow$ collective models to describe smooth systematic variation, like binding energy (LDM, RLD, vibration/rotation, deformation)
- Single-Particle motion $\rightarrow$ Fermi-gas model (the other extreme), nuclear shell model
- Single particle in a mean field, $L - S$ coupling, shell structure
- Mean-field potentials: harmonic oscillator (spherical/cartesian), Woods-Saxon, etc. for a single nucleon
- Deformed single-particle potential $\rightarrow$ Nilsson model
- Many-particle shell model, pairing correlations
- Predicted quantities: total binding energies, level energies, transition rates, compared with experiments
Literature

Books
- Bertulani Ch 5
- K.S. Krane
- Bohr & Mottelson, Nuclear Structure, vols 1 & vol 2
- K. Heyde, Basic Ideas and Concepts in Nuclear Physics, Ch. 7-10

Journals etc.
- B. A. Brown: Lecture notes 2011
- Advances in Nuclear Physics
- Annual Review of Nuclear Particle Science
- Reviews of Modern Physics
- MIT, TALENT, youtube, vimeo
Historically the first model to describe nuclear properties, based on observation that nuclear forces exhibit saturation properties:

- The binding energy per nucleon $\frac{BE(A, Z)}{A}$ relativley constant,
- instead of the form $A(A - 1)/2$ predicted by a non-saturated 2-body nucleon-nucleon interaction
- Also, the nucleus presents a low compressibility,
- and a well defined nuclear surface.
- liquid drop model taken as a fully classical model, cannot be extrapolated too far in the atomic nucleus:
- spikes on $\frac{BE(A, Z)}{A}$ appear at $A = 4n$, presenting a favoured $\alpha$-particle like structure
• average distance of nucleons much larger than in a classical fluid (2-body potential min. at 0.7 fm) because they are fermions $\rightarrow$ Fermi liquid. Collisions very rare. Resulting system is a weakly interacting Fermi gas. No allowed levels to scatter to (Pauli exclusion). Mean free path of the order of nuclear diameter.

• Saturation in binding results is almost constant $BE(A, Z)/A \approx 8$ MeV independent of $A, Z$. 

![Graph showing binding energy per nucleon vs mass number A]
1. The volume term expresses the fact that nuclear force is saturated with the main contribution as

\[ \text{BE}(A, Z)_V = a_V A. \] (147)

2. The surface effect: nucleons close to surface feel less binding, the correction being \( \propto 4\pi R^2 \),

\[ \text{BE}(A, Z)_S = -a_S A^{2/3}. \] (148)

3. Coulomb effect, a charge of \( Ze \) present within nuclear volume. For a homogeneously charged liquid drop with sharp radius \( R \) and density

\[ \rho_c = \frac{Ze}{\frac{4}{3}\pi R^3} \] (149)
Use a classical argument:

The Coulomb energy needed to add a spherical shell, to the outside of the sphere with radius $r$, to give an increment $dr$ becomes

$$U'_c = \frac{1}{4\pi \epsilon_0} \int_0^R \frac{4\pi r^3 \rho_c 4\pi r^2 \rho_c}{r} dr$$  \hspace{1cm} (150)

Using the above charge density, the integral becomes

$$U'_c = \frac{3}{5} \frac{Z^2 e^2}{R} \frac{1}{4\pi \epsilon_0}$$  \hspace{1cm} (151)
But wait, the protons’ spurious self-energy must be subtracted:

Using the proton smeared charge density

$$\rho_p = \frac{e}{\frac{4}{3} \pi R^3}$$  \hspace{1cm} (152)

and a self-Coulomb energy, for the $Z$ protons, as

$$U''_c = \frac{3}{5} \frac{Ze^2}{R} \frac{1}{4\pi \epsilon_0}$$  \hspace{1cm} (153)

the total Coulomb energy correction becomes

$$U_c = U'_c - U''_c = \frac{3}{5} \frac{Z(Z - 1)e^2}{R} \frac{1}{4\pi \epsilon_0} = a_c Z(Z - 1) A^{-1/3}$$  \hspace{1cm} (154)

The three contributions together then give

$$BE(A, Z) = a_V A - a_S A^{2/3} - a_c Z(Z - 1) A^{-1/3},$$  \hspace{1cm} (155)

or, per nucleon,

$$BE(A, Z)/A = a_V - a_S A^{-1/3} - a_c Z(Z - 1) A^{-4/3}.$$  \hspace{1cm} (156)
Left panel: Squares show experimental BE/A, curve the contribution of volume, surface and Coulomb terms.

Right panel: Contribution of separate terms, surface term largest at small $A$, Coulomb correction (estimated by $Z = A/2$) largest for heavy nuclei (many protons, $A^{2/3}$ effect).
Non-spherical shape

- Surface and Coulomb energy corrections will change.
- Lowest deformation multipoles via the \((\theta, \varphi)\) expansion

\[
R = R_0 \left[ 1 + \alpha_2 P_2(\cos \theta) + \alpha_4 P_4(\cos \theta) \right] \Rightarrow \tag{157}
\]

- Correction functions for surface energy \(g(\alpha_2, \alpha_4)\) and for Coulomb energy \(f(\alpha_2, \alpha_4)\) appear in the expression for

\[
BE(A, Z)/A = a_V - a_S g(\alpha_2, \alpha_4) A^{-1/3} - a_c f(\alpha_2, \alpha_4) Z (Z - 1) A^{-4/3}. \tag{158}
\]

- Example: ellipsoidal deformation with major axis

\[
a = R(1 + \varepsilon), \quad \left( \varepsilon = \sqrt{1 - b^2/a^2} \right), \tag{159}
\]

and the minor axis

\[
b = R(1 + \varepsilon)^{-1/2}, \tag{160}
\]

with volume \(V = \frac{4}{3} \pi ab^2 \approx \frac{4}{3} \pi R^3\).
• In terms of parameter of deformation $\varepsilon$,
• surface and Coulomb energy terms become

$$
E_S = a_S A^{2/3} \left(1 + \frac{2}{5} \varepsilon^2\right)
$$

$$
E_C = a_c Z (Z - 1) A^{-1/3} \left(1 - \frac{1}{5} \varepsilon^2\right).
$$

(161)

• The total energy change, due to deformation

$$
\Delta E = \Delta E_S + \Delta E_C = \varepsilon^2 \left[\frac{2}{5} A^{2/3} a_S - \frac{1}{5} a_c Z (Z - 1) A^{-1/3}\right].
$$

(162)

• If $\Delta E > 0$, the spherical shape is stable if $Z^2/A < 49$, as can be “proved” by simplifying $Z(Z - 1) \approx Z^2$ and using the best-fit values (Wapstra 1971), $a_S=17.2$ MeV and $a_C=0.70$ MeV.
Deformation, a doorway to fission?

Dashed line: potential energy versus nuclear distance between two nuclei. The point at $\varepsilon = 0$ corresponds to the spherical nucleus. For large separation, calling $r = R_1 + R_2$ (with $R_1$ and $R_2$ the radii of both fragments), the energy varies according to the Coulomb energy

$$U_C = \frac{Z_1 Z_2 e^2}{4\pi \varepsilon_0 r}$$

Deformation complicates the precise evaluation of the full total potential energy and requires complicated fission calculations.
Symmetry energy, pairing and shell corrections

The specific nucleon (Dirac-Fermi statistics) properties of the nuclear interior modify a number of results. Manifestations of the Pauli principle governing the occupation of the single-particle orbitals in the nuclear, average field and the nucleonic residual interactions that try to pair-off identical nucleons to \(0^+\) coupled pairs.

- **Symmetry energy:**
  Binding energy largest when \(Z = N = A/2\), and any repartition \(N = (A/2) + \nu\), \(Z = (A/2) - \nu\) involves lifting nucleons from occupied to empty orbitals.

- Energy loss due to repartition (\(\Delta\) is the inter-orbital energy difference):

  \[
  \Delta E_{\text{binding}} = \nu \left( \Delta \times \frac{\nu}{2} \right) = \frac{1}{8} (N - Z)^2 \Delta
  \]

  with \(\nu = (N - Z)/2\).

- The energy spacing between single-particle states scales as \(A^{-1}\)
The final result, expressing the loss of symmetry energy due to the Pauli effect which blocks the occupation of those levels that already contain two identical nucleons, becomes

$$BE(A, Z) = a_V A - a_S A^{2/3} - a_C Z (Z - 1) A^{-1/3} - a_A (A - 2Z) A^{-1}. \quad (164)$$

Better understanding from Fermi-gas model (later derivation of the coefficient $a_A$)

Pairing energy contribution:
Nucleons preferentially form pairs coupled to $J = 0$, resulting in a pairing energy correction $\delta = a_P A^{1/2}$:

$$\Delta E_{\text{pair}} = \begin{cases} 
+\delta & \text{(even-even)} \\
0 & \text{(odd-even)} \\
-\delta & \text{(odd-odd)} 
\end{cases} \quad (165)$$
After all pieces are put together, we have finally: **semi-empirical mass equation**

(Bethe-Weizsäcker formula)

\[
BE(A, Z) = a_V A - a_S A^{2/3} - a_C Z(Z - 1) A^{-1/3} - a_A (A - 2Z)^2 A^{-1} \begin{cases} 
+\delta \\
0 \\
-\delta 
\end{cases} \tag{166}
\]

The parameters fitted to experimental values by Wapstra (1971) and more recently by Bertsh *et al*:

<table>
<thead>
<tr>
<th>Param. Values from fit by</th>
<th>Wapstra</th>
<th>Bertsh <em>et al</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>[MeV]</td>
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<tr>
<td>(a_V)</td>
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<td>15.74063</td>
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<tr>
<td>(a_S)</td>
<td>18.34</td>
<td>17.61628</td>
</tr>
<tr>
<td>(a_C)</td>
<td>0.71</td>
<td>0.71544</td>
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<tr>
<td>(a_A)</td>
<td>23.21</td>
<td>23.42742</td>
</tr>
<tr>
<td>(a_p)</td>
<td>12</td>
<td>12.59898</td>
</tr>
</tbody>
</table>

Note that Bertulani defines the antisymmetry term differently which implies that \(a_A(\text{“Bertulani”}) = 4 \times a_A(\text{“ours”})\)

- but you can make your own parameter fit to the most recent atomic masses [G. Audi *et al*, Chinese Physics *36*, 1603, 2012]!
- The smooth cross behaviour is reproduced nicely, but large deviations occur near **closed shells**:
Figure 28: Nuclear (liquid drop) masses and the deviations with respect to the nuclear data and this, as a function of proton and neutron number. The shell closure effects are shown most dramatically (Myers 1966).

The relation between the nuclear binding energy and nuclear mass is

$$BE(A, Z)/c^2 = M_N(A, Z) - ZM_p - (A - Z)M_n$$  \hspace{1cm} (167)
In case the difference between atomic and nuclear mass is important, use the relation
\[
M_{\text{Nucl.}}(A, Z) = M_{\text{Atom}}(A, Z) - Z \times m_{\text{el.}} + B_{\text{el}}(Z)/c^2, \tag{168}
\]
where the total binding energy of the atomic electrons can be approximated by
\[
B_{\text{el}}(Z) = 14.4381Z^{2.39} + 1.55468 \times 10^{-6}Z^{5.35}[\text{eV}]. \tag{169}
\]
The most precisely known masses (2012) are listed in the table next page for reference. Who knows, when you need them!

In the literature, the atomic mass is often given as mass excess (either in mass units u or in eV\textsubscript{90})
\[
\Delta = M - A
\]
The reason is that the mass \(M(A, Z)\) rounded to nearest integer equals \(A\).

\[1\] For high-accuracy work, it’s important to specify what definition of the unit volt, V, is used when converting mass units [u] to energy units [eV]. The latest Atomic Mass Evaluation (2012), recommends the maintained volt \(V_{90}\), that is traceable to primary standards. \(1 \text{ uc}^2 = 931.494.0023 \pm 0.0007 \text{ keV}_{90}\)
<table>
<thead>
<tr>
<th>Element</th>
<th>Mass excess (keV\text{\textsubscript{90}})</th>
<th>Mass excess uncertainty</th>
<th>Atomic mass (\mu u)</th>
<th>Atomic mass uncertainty</th>
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<td>8 071.317 144</td>
<td>0.000 458</td>
<td>1 008 664.915 850</td>
<td>0.000 491</td>
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<td>0.000 059</td>
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<td>0.000 211</td>
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<tr>
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<td>0.000 188</td>
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<tr>
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<td>0.000 599</td>
<td>15 000 108.898 884</td>
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<tr>
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<td>- 4 737.001 374</td>
<td>0.000 160</td>
<td>15 994 914.619 566</td>
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<tr>
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<tr>
<td>\textsuperscript{18}O</td>
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<td>0.000 707</td>
<td>17 999 159.612 858</td>
<td>0.000 758</td>
</tr>
<tr>
<td>\textsuperscript{19}F</td>
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<td>0.000 860</td>
<td>18 998 403.162 727</td>
<td>0.000 923</td>
</tr>
<tr>
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<td>0.000 410</td>
<td>27 976 926.534 649</td>
<td>0.000 440</td>
</tr>
<tr>
<td>\textsuperscript{29}Si</td>
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<td>0.000 487</td>
<td>28 976 494.664 901</td>
<td>0.000 523</td>
</tr>
<tr>
<td>\textsuperscript{31}P</td>
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<td>0.000 655</td>
<td>30 973 761.998 417</td>
<td>0.000 702</td>
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<tr>
<td>\textsuperscript{32}S</td>
<td>- 26 015.533 531</td>
<td>0.001 316</td>
<td>31 972 071.174 408</td>
<td>0.001 412</td>
</tr>
</tbody>
</table>
Figure 29: Nuclide chart of all the nuclei with experimentally known atomic masses (2012). The blue curve shows the *valley of stability* as predicted by the LDM. The horizontal lines show the *magic proton numbers* at $Z = 2, 8, 20, 28, 50, 82$ (©P.O.T. 14.11.2017).
Nuclear mass equation from the Bethe-Weizsäcker formula:

\[
M_{\text{nucl.}}(\frac{4}{A}X_N)c^2 = ZM_p c^2 + (A - Z)M_n c^2 - a_v A + a_s A^{2/3} + a_C Z(Z - 1)A^{-1/3} + a_A(A - 2Z)^2 A^{-1} \begin{cases} -\delta \\
0 \\
+\delta \end{cases}
\]

(170)

representing a quadratic equation in the proton number \( Z \), i.e.,

\[
M_{\text{nucl.}}(\frac{4}{A}X_N)c^2 = xA + yZ + zZ^2 \begin{cases} +\delta \\
0 \\
-\delta \end{cases}
\]

(171)

The most stable nucleus for each \( Z \) is then approximately (show!)

\[
Z_{\text{min}}(A) = \frac{A/2}{1 + \frac{1}{4}a_C/a_A A^{2/3}} = \frac{A/2}{1 + 0.0077A^{2/3}}.
\]

(172)
Several interesting results can be derived

- Parabolic behaviour of nuclear masses for a given $A$: (i) Members of an odd-$A$ chain can decay—towards a single stable nucleus—either by $\beta^-$ or $\beta^+/EC$, but not both. (ii) For even-$A$ nuclei, both even-even and odd-odd nuclei can occur and (because of the $\delta$ value) two parabolae are implied by the mass equation. In many cases, more than one stable element results (double-beta decay!) There are even cases with three ‘stable’ elements which depend on the specific curvature of the parabola and the precise location of the integer $Z$ values:
Figure 30: Mass parabola for $A = 101$
Constraints on the range of possible $A, Z$ values and stability against radioactive decay processes can be determined. Spontaneous $\alpha$-decay ($S_\alpha = 0$) follows from the equation

$$BE(\frac{A}{Z}X_N) - \left[BE(\frac{A-4}{Z-2}Y_{N-2}) + BE(\frac{4}{2}He_2)\right] = 0.$$  

(173)

The condition $S_n = 0$ ($S_p = 0$), indicates the borderline where a neutron (proton) is no longer bound in the nucleus. It’s called the neutron (proton) drip line.

The energy released in nuclear fission, in the simple case of symmetric fission of the element $(A, Z)$ into two nuclei $(A/2, Z/2)$ is

$$E_{\text{fission}} = M_{\text{nucl.}}(\frac{A}{Z}X_N)c^2 - 2M_{\text{nucl.}}(\frac{A/2}{Z/2}Y_{N/2})c^2$$

$$= [-4.78A^{2/3} + 0.26Z^2A^{-1/3}]c^2$$  

(174)

and becomes positive near $A \approx 75$, reaching a value of about 185 MeV for $^{236}U$.\(^2\)

\(^2\)For these fission products, neutron-rich nuclei are obtained which will decay by the emission of $n \to p + e^- + \bar{\nu}_e$. So, a fission process is a good generator of $\bar{\nu}_e$, antineutrinos of the electron type; the fusion process on the other hand will mainly give rise to $\nu_e$ electron neutrino beta-decay transitions.
Two-neutron separation energies

- Nuclear masses and a derived quantity, $S_{2n}$, may reveal the presence of extra correlations on top of a smooth liquid-drop behaviour:

$$S_{2n}(A, Z) = BE(A, Z) - BE(A - 2, Z)$$  \hspace{1cm} (175)

- Can be written as

$$S_{2n} \approx 2(a_V - a_A) - \frac{4}{3} a_S A^{-1/3} + \frac{2}{3} a_C Z(Z - 1) A^{-4/3} + 8a_A \frac{Z^2}{A(A - 2)}$$

where the surface and Coulomb terms are only approximated expressions.

- Insert $Z_0$, that maximizes the binding energy for each given $A$ (this is the definition for the valley of stability),

$$Z_0 = \frac{A/2}{1 + 0.0077 A^{2/3}},$$  \hspace{1cm} (176)

to obtain, for large values of $A$,

$$S_{2n} = 2(a_V - a_A) - \frac{4}{3} a_S A^{-1/3} + \left(8a_A + \frac{2}{3} a_C A^{2/3}\right) \frac{1}{4 + 0.06 A^{2/3}}$$
- Behaviour of $S_{2n}$ along the valley of stability for even-even nuclei, together with the experimental data.
- The experimental data correspond to a range of $Z$ between $Z - 1$ and $Z - 2$.
- Overall decrease and the specific mass dependence are well contained within the LDM.
- How well the experimental two-neutron separation energy, through a chain of isotopes, is reproduced in LDM.
- Besides the sudden variations near mass number $A = 90$ (presence of shell closure at $N = 50$) and near mass number $A = 140$ (presence of the shell closure at $N = 82$), the specific mass dependence for a series of isotopes comes closer to specific sets of straight lines.
- LDM is able to describe the almost linear behaviour of $S_{2n}$ for series of isotopes; expand the various terms (volume, surface, etc.) in terms of $X = A - A_0$ and $\epsilon = X/A_0$ near a fixed $A$ to see their separate contributions. It turns out that the slope is determined mainly by the asymmetry term.
Fermi gas model

- Approximate distribution for kinetic energy of nucleons confined to the nuclear volume \( V = \frac{4}{3} \pi R^3 \) with density \( \rho = \frac{A}{V} \).
- Strong spatial confinement results in widely spaced energy levels with only the lowest being occupied, except for very high energies.
- Appropriate model, degenerate Fermi gas, up to excitation of \( \sim 10 \) MeV.
- Under conditions present in nuclear matter but approximates also heavy nuclei.
- Assume lowest energy state allowed by Pauli exclusion principle.
- Start with symmetric nuclear matter, \( Z = N \).
- At low excitation energy (temperature \( T = 0 \) limit), levels pairwise filled up to Fermi level, beyond that levels fully unoccupied: *degenerate Fermi gas*.
- At high temperature, occupancy partially redistributed.
- At very high temperature, all levels partially filled.
Neutrons (left) and protons (right) in nuclear square well. Degenerate $T = 0$ Fermi gas.

Level occupancy at $T = 0$ (shaded), higher $T > 0$ and very high $T \gg 0$ temperatures.

Density of states for free Fermi gas confined to volume $V$: \[^3\]

$$n = \frac{2V}{(2\pi\hbar)^3} \int_0^{p_F} d^3 p = \frac{V p_F^3}{3\pi^2 \hbar^3},$$  \hspace{1cm} (177)

[^3]: See BAB for an alternative derivation with a cube of side $L$. 
• resulting in the *Fermi momentum*

\[ p_F = \hbar \left(3\pi^2 \frac{n}{V}\right)^{1/3} = \hbar \left(\frac{3}{8\pi} \frac{n}{V}\right)^{1/3}, \]  

(178)

where \( n/V \) is the density contributing to a quantum mechanical pressure.

• Apply to atomic nucleus to estimate the depth of the potential well, \( U_0 \):

\[ p_{F,n} = \frac{\hbar}{r_0} \left(\frac{9\pi N}{4A}\right)^{1/3}, \quad p_{F,p} = \frac{\hbar}{r_0} \left(\frac{9\pi Z}{4A}\right)^{1/3}, \]  

(179)

and for self-conjugate nuclei, \( N = Z = A/2 \):

\[ p_{F,n} = p_{F,p} = \frac{\hbar}{r_0} \left(\frac{9\pi}{8}\right)^{1/3} \approx \frac{300}{r_0} \text{ MeV}/c. \]  

(180)

• The corresponding Fermi (kinetic) energy with \( r_0 = 1.2 \text{ fm} \) becomes

\[ E_F = \frac{p_{F,n}^2}{2m} = \frac{p_{F,n}^2}{2m} \approx 33 \text{ MeV}. \]  

(181)
• This energy corresponds to the kinetic energy of the highest occupied orbit (smallest binding energy).

• Given the average binding energy $B = 8 \text{ MeV}$, we can make a good estimate of the nuclear well depth of $U_0 \approx 41 \text{ MeV}$.

• In more realistic calculations, values of this order are obtained.

• The average kinetic energy per nucleon (for non-relativistic motion) is

$$\langle E \rangle = \frac{\int_0^{p_F} E_{\text{kin}} d^3p}{\int_0^{p_F} d^3p} = \frac{3}{5} \frac{p_F^2}{2m} = \frac{3}{5} E_F \approx 20 \text{ MeV}$$

(182)

• In the degenerate Fermi-gas model, even at zero excitation energy, a large amount of ‘zero-point’ energy is present and, owing to the Pauli principle, a quantum-mechanical ‘pressure’ results.

• Given a nucleus with $Z$ protons and $N$ neutrons, the total, average kinetic energy becomes

$$\langle E(A, Z) \rangle = Z \langle E \rangle_p + N \langle E \rangle_n = \frac{3}{10m} \left( Zp_{F,p}^2 + Np_{F,n}^2 \right)$$

$$= \frac{3}{10m} \frac{\hbar^2}{r_0^2} \left( \frac{9\pi}{4} \right)^{2/3} \frac{\left( N^{5/3} + Z^{5/3} \right)}{A^{2/3}}$$
Fermi gas: Nuclear symmetry potential

Expand the formula of the average energy at $N = Z = A/2$ to derive an expression of the symmetry energy and a value for the constant $a_A$:

- Set $Z - N = \epsilon$ to have $Z = A/2(1 + \epsilon/A)$; $N = A/2(1 - \epsilon/A)$ and with $\epsilon/A \ll 1$ obtain (using binomial expansion):
  \[
  \langle E(A, Z) \rangle = \frac{3}{10m} \frac{\hbar^2}{r_0^2} \left( \frac{9\pi}{4} \right)^{2/3} A \left( \frac{1}{2} \right)^{5/3} \frac{(1 - \epsilon/A)^{5/3} + (1 + \epsilon/A)^{5/3}}{A^{2/3}} \]
  \[
  = \frac{3}{10m} \frac{\hbar^2}{r_0^2} \left( \frac{9\pi}{8} \right)^{2/3} \left( A + \frac{5}{9} \frac{(N - Z)^2}{A} + \ldots \right)
  \]
  
  - The first term is proportional to $A \leftrightarrow$ volume energy.
  - The next term has exactly the form of the symmetry energy $A, Z$-dependent term in the Bethe-Weizsäcker mass equation.
  - Inserting the values of the various constants $m, \hbar, r_0, \pi$, we obtain the result

\[
\langle E(A, Z) \rangle = \langle E(A, Z = A/2) \rangle + \Delta E_{\text{symm}}
\]

\[
\Delta E_{\text{symm}} \simeq 11 \text{ MeV}(N - Z)^2 A^{-1}.
\]

(183)
The numerical result is about one-half the value obtained earlier (or fitted). The difference arises from the fact that the well depth $U_0$ also depends on the neutron excess $N − Z$.

Higher-order terms with a dependence $(N − Z)^4/A^3, . . .$ naturally derive from the more general $\langle E(A, Z) \rangle$ value.

- The Fermi-gas model can be applied successfully to various different problems:
  - Stability of degenerate electron gas: white dwarfs stars
  - Stability of degenerate neutron gas: description of neutron stars
  - Balancing the gravitational energy with the quantum-mechanical Fermi-gas energy to obtain equation of state for the neutron star. (Bertulani 12.11, Heyde 8.3)
The Nuclear Shell Model

Single-particle motion cannot be completely replaced by a collective approach where the dynamics is contained in collective, small amplitude vibrations and/or rotations of the nucleus as a whole.

A simple independent-particle approach required by the Fermi-gas model does not contain enough detailed features of the nucleon-nucleon forces, active in the nucleus.

Experimental evidence for nuclear shell structure is seen as deviations from otherwise rather smooth behaviour of various quantities as a function of $N$ and $Z$:

- Neutron and proton separation energies $S_n$ and $S_p$ and two-neutron separation energies $S_{2n}$ (you can make your own plots!),
- Extra stability at neutron number 8, 20, 28, 50, 82, and 126, or at proton number 8, 20, 28, 50, and 82. (no $Z = 126$ element known!)
- Excitation energy of the first $2^+$ state in even-even nuclei,
- Reduced transition probabilities of electric quadrupole transitions, $B(E2; 0^+ \rightarrow 2^+) \uparrow$, in even-even nuclei.
The left panel shows the excitation energies of the first $2^+$ states in even-even nuclei, the right panel the reduced transition probabilities $B(E2; 0^+ \rightarrow 2^+)$.\(^4\)

\(^4\)The vertical lines show the location of magic proton numbers at $Z = 8, 20, 28, 50, \text{ and } 82.$
As a function of neutron number: (a) $E_{\text{ex}}(2^+)$, (b) $B(\text{E}2; 0^+ \rightarrow 2^+)$, (c) Deformation parameter $\beta / \beta_{\text{sp}}$ (d) EWSR(II) = Energy Weighted Sum rule
As a function of proton number: (a) $E_{\text{ex}}(2^+)$, (b) $B(E2; 0^+ \rightarrow 2^+)$, (c) Deformation parameter $\beta / \beta_{\text{sp}}$ (d) EWSR(II) = Energy Weighted Sum rule
Before jumping into the details of the nuclear shell model, a few words on collective motion of nucleons and on the criteria to collectivity.

Nuclear collective motion is fairly distinctive in physics. There is some similarity with the phenomena of oscillations of a classical liquid drop and with electron gas oscillations, but there are also essential differences. Unlike the liquid drop, the nucleus is a quantum system, and its collective parameters differ considerably from hydrodynamic values. The most obvious difference with the plasma is that the nucleus is a relatively small system, so that its collective modes are characterized by angular rather than linear momenta, and also by parity. A further difference arises from the existence of two types of nuclear particle. These may oscillate in phase or out of phase. In the language of isospin, we say that the two types of oscillation are isoscalar \((T = 0)\) and isovector \((T = 1)\).
Nuclear collective modes are thus classified by $\mathcal{L} = (T, L, \pi)$ (isospin, angular momentum, parity). We refer to $\mathcal{L}$ as the multipole order. Experimentally the only well-established modes are those with parity $\pi = (-1)^L$, i.e., in electromagnetic terminology, “electric multipoles” such as E3 signifying $L = 3, \pi = -$. For such modes it is convenient to define “multipole operators”:

$$Q_{TLM} = \begin{cases} \sum_i r_i^L Y_{LM}(\Omega_i) & \text{for } T = 0 \\ \sum_i \tau_3 i r_i^L Y_{LM}(\Omega_i) & \text{for } T = 1 \end{cases}$$

(184)

Note that $Q_{0LM}$ is the “mass multipole operator" and the “charge multipole operator” (which refers to protons only) is

$$Q_{ELM} = \frac{1}{2} (Q_{0LM} - Q_{1LM})$$
In surveying the data on nuclei in a given mass region, there are two basic criteria used for deciding whether collective phenomena occur:

1. Regularities in spectra characteristic of collective modes. The simplest are the uniform spacing of the vibrator and the \( J(J + 1) \)-type spectrum of the rotator. A less decisive criterion is the systematic occurrence of a given type of level in different nuclei.

2. Strong electromagnetic matrix elements, either diagonal (e.g., E2 moments) or non-diagonal (e.g., EL transitions). By “strong” here, we mean at least several times larger than a typical value for a single proton. The matrix element for a proton transition between a particle state of orbital angular momentum \( L \) and one of \( L = 0 \) is:

\[
\langle LM | Q_{0LM} | 00 \rangle = \frac{1}{\sqrt{4\pi}} \int_0^\infty \left[ u_L(r) r^L u_0(r) \right] r^2 \, dr
\]

where the \( u \)'s are radial wavefunctions. For a rough estimate, we may replace the integral by the \( L^{th} \) moment of the density distribution \( \rho(r) \) of the nucleus concerned:

\[
\langle r^L \rangle = \int r^L \rho(r) \, dr / \int \rho(r) \, dr.
\]
An alternative version of criterion (2) is that the transition exhausts at least a fair fraction (say $\gtrsim 5$ per cent) of a sum rule. There are two sum rules that are relevant; in obvious notations, these are

$$\sum_n |\langle n|Q_{TL0}|0\rangle|^2 = \langle 0|(Q_{TL0})^2|0\rangle \equiv S_{NEW}^{TL}$$

$$\sum_n (E_n - E_0)|\langle n|Q_{TL0}|0\rangle|^2 = \frac{1}{2} \langle 0|[Q_{TL0}, [H, Q_{TL0}]]|0\rangle \equiv S_{EW}^{TL}$$

We call these the non-energy-weighted (NEW) and energy-weighted (EW) sum rules. The only sum that can be evaluated exactly (i.e., without reference to a model) is the EWS for $T = 0$:

$$S_{EW}^{0L} = \frac{\hbar^2 A}{8\pi M} L(2L + 1) \langle r^{2L-2}\rangle$$

The only assumption is that $H$ contains no explicitly velocity-dependent forces (exchange forces are permitted). Strictly, this value is obtained when the ground state has spin 0, but any correction for non-zero spin is small. It is fortunate that most observed modes appear to be (or are assumed to be) $T = 0$. 


• Assuming a uniformly charged spherical nucleus with a sharp radius $R_0 = r_0 A^{1/3}$, the energy-weighted sum for the E2 multipole

$$S_I \equiv S_{EW}^{02} = \sum_n (E_n - E_0) B(E2) \uparrow = 30e^2 \frac{\hbar^2}{8\pi m} AR_0^2$$  \hspace{1cm} (185)

where $m$ is the nucleon mass.

• The isoscalar part of the full sum is given by

$$S_{II} \equiv S(I)(\frac{Z}{A})^2.$$  \hspace{1cm} (186)

• The percentage fraction of $E \times B(E2; 0^+ \rightarrow 2^+) \uparrow$ of $S_{II}$, for only the first $2^+$ state only in even-even nuclei is shown in the above figures.

• The deformation parameter (there are several definitions in the literature!) shown in the figures is related to the $B(E2) \uparrow$ by

$$\beta = \frac{4\pi}{3} ZR_0^2 \left[ B(E2) \uparrow / e^2 \right]^{1/2},$$  \hspace{1cm} (187)

with the corresponding single-particle (Weisskopf) estimate $\beta_{sp} = 1.59/Z$. 
The Nuclear Shell Model
The Nuclear Shell Model

Literature

- Bertulani, Ch 5.4,
- BAB: Ch 8, Ch 9 *The harmonic oscillator*, Ch 10.1 *The Woods-Saxon potential*, Ch 11, *The general many-body problem for fermions*
- As an introduction to the subject, read BAB: Ch 8 *Overview of the nuclear shell model*
- A. deShalit and I. Talmi
- A. Bohr and B. Mottelson, Nuclear Structure, vols. I & 2
The 1st approximation to the hard many-nucleon problem.

Nucleons move in a central average potential, but otherwise independently ⇔

Single-particle shell model or extreme shell model:

Central potential $U(r)$ plus spin-orbit interaction of the form $f(r)\vec{l} \cdot \vec{s}$

Choice of central potential: 1) square-well, 2) simple harmonic oscillator (SHO), 3) Woods-Saxon, depending on which one is the most convenient (analytical, symmetry, etc. properties)

Woods-Saxon approximates the nuclear density distribution. Nuclear force of short range and potential should follow the density.

Non-central potential: Nilsson model, assumes the potential to be deformed from the beginning
Central potential: Square well

- Separate the 3-d Schrödinger equation in spherical coordinates
- Spherical harmonics solve the angular part, radial equation depends on the potential
- For the square-well potential model

\[
U(r) = \begin{cases} 
-U_0, & r \leq R \\
\infty, & r > R 
\end{cases} \tag{188}
\]

the corresponding Schrödinger one-particle equation becomes

\[
\left[-\frac{\hbar^2}{2m} \Delta + U(r)\right] \Phi(\vec{r}) = E \Phi(\vec{r}), \tag{189}
\]

with

\[
\Delta = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\hat{l}^2}{\hbar^2} \frac{1}{r^2},
\]

\(\hat{l}^2\) denotes the angular momentum operator, with eigenfunctions \(Y_l^m(\theta, \varphi)\) and corresponding eigenvalues \(l(l + 1)\hbar^2\). Separating the wavefunction as

\[
\Phi(\vec{r}) = R_{nl}(r) Y_l^m(\theta, \varphi) \tag{190}
\]
the corresponding radial equation becomes

\[
\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left[ \frac{2m}{\hbar^2} (E - U(r)) - \frac{l(l + 1)}{r^2} \right] R(r) = 0.
\] (191)

The solutions become the solutions of the Bessel differential equation (for the regular ones which we concentrate on)

\[
R_{nl}(r) = \frac{A}{\sqrt{kr}} J_{l+1/2}(kr).
\] (192)

These solutions are regular at the origin (the spherical Bessel functions of the first kind) with

\[
j_l(kr) = \sqrt{\frac{\pi}{2kr}} J_{l+1/2}(kr).
\] (193)

The energy eigenvalues

\[
R_{nl}(R) = 0 \quad \text{or} \quad J_{l+1/2}(k_{nl}R) = 0.
\] (194)

For these \(k_{nl}R\) coincides with the zeros of the spherical Bessel function,

\[
E_{nl} = \frac{\hbar^2 k_{nl}^2}{2m} = \frac{\hbar^2 X_{nl}^2}{2mR^2}
\] (195)

with \(X_{nl} = k_{nl}R\).
The harmonic oscillator potential

- The radial problem for a spherical, harmonic oscillator potential can be solved either
- (i) Using a Cartesian basis

\[ U(r) = -U_0 + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2). \]  

(196)

The three \((x, y, z)\) specific one-dimensional oscillator eigenvalue equations become

\[
\left[ \frac{d^2}{dx^2} + \frac{2m}{\hbar^2} \left( E_1 + \frac{U_0}{3} - \frac{1}{2}m\omega^2 x^2 \right) \right] \phi_1(x) = 0 \]  

(197)

with

\[ E = E_1 + E_2 + E_3. \]  

(198)

The three eigenvalues are then

\[ E_i = \hbar\omega(n_i + 1/2) - U_0/3, \quad i = 1, 2, 3 \]  

(199)

or

\[ E = \hbar\omega(N + 3/2) - U_0, \]

\((N = n_1 + n_2 + n_3 \text{ with } n_1, n_2, n_3 \text{ three positive integer numbers } 0, 1, \ldots).\)
The wavefunctions for the one-dimensional oscillator are the Hermite polynomials, characterized by the radial quantum number $n_i$, so

$$\phi_1(x) = N_1 \exp \left(-\frac{m\omega}{2\hbar} x^2\right) H_{n_1}(\nu x), \quad \left(\nu = \sqrt{\frac{m\omega}{\hbar}}\right)$$

(200)

and

$$\Phi(x, y, z) = \phi_1(x)\phi_2(y)\phi_3(z),$$

(201)

$N_1, N_2, N_3$ are normalization coefficients.

(ii) Separate the radial variable $r$ from the angular $(\theta, \varphi)$ ones. In this case, the radial equation reduces to

$$\frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left(\frac{2mE}{\hbar^2} + \frac{2m}{\hbar^2} U_0 - \frac{m^2\omega^2}{\hbar^2} r^2 - \frac{l(l+1)}{r^2}\right) R(r) = 0.$$
The normalized radial solutions are the Laguerre polynomials

\[ R_{nl}(r) = N_{nl}(\nu r)^l \exp \left( \frac{-\nu^2 r^2}{2} \right) L_{n-1}^{l+1/2}(\nu^2 r^2), \]  

(203)

and \( \nu = \sqrt{m\omega/\hbar} \), describing the oscillator frequency. The Laguerre polynomials (Abramowitz and Stegun 1964) are given by the series

\[ L_{n-1}^{l+1/2}(x) = \sum_{k=0}^{n-1} a_k^l (-1)^k x^k. \]  

(204)

In BAB there are given expressions that show all explicitly. Also see Bertulani.

\( (2(n-1) + l) \) can be identified with the major oscillator quantum number \( N \) we obtained in the Cartesian description. The total wavefunction then corresponds to energies

\[ E_N = (2(n-1) + l)\hbar\omega + \frac{3}{2} \hbar\omega - U_0. \]  

(205)

For a specific level \( (n, l) \) there exists degeneracy relative to the energy characterized by quantum number \( N \), i.e. we have to construct all possible \( (n, l) \) values such that

\[ 2(n-1) + l = N. \]  

(206)
Table 7: 3-d simple harmonic oscillator energies.

<table>
<thead>
<tr>
<th>N</th>
<th>$E_N(\hbar\omega)$</th>
<th>(n, l)</th>
<th>$\sum_{nl} 2(2l + l)$</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3/2</td>
<td>1s</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>5/2</td>
<td>1p</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>7/2</td>
<td>2s, 1d</td>
<td>12</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>9/2</td>
<td>2p, 1f</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>11/2</td>
<td>3s, 2d, 1g</td>
<td>30</td>
<td>70</td>
</tr>
<tr>
<td>5</td>
<td>13/2</td>
<td>3p, 2f, 1h</td>
<td>42</td>
<td>112</td>
</tr>
<tr>
<td>6</td>
<td>15/2</td>
<td>4s, 3d, 2g, 1i</td>
<td>56</td>
<td>168</td>
</tr>
</tbody>
</table>

An alternative parametrization uses HO length parameter $b = 1/\nu = \sqrt{\hbar/m\omega}$. The radial wavefunctions are then given as\(^5\)

\[
R_\alpha(r) = \sqrt{\frac{2^{l-n_r+2}(2l + 2n_r + 1)!!}{\sqrt{\pi}(n_r)!b^{2l+3}[(2l + 1)!!]^2}} r^{l+1} e^{-r^2/2b^2} \\
\times \sum_{k=0}^{n_r} \frac{(-1)^k 2^k (n_r)!(2l + 1)!!}{k!(n_r - k)!(2l + 2k + 1)!!} (r/b)^{2k}
\]

\(^5\)Note that in this parametrization, $n_r = n - 1$. 
Figure 31: Illustration of the harmonic oscillator radial wavefunctions $rR_{nl}(r)$. The upper part shows, for given $l$, the variation with radial quantum number $n$ ($n = 1$: no nodes in the interval $(0, \infty)$). The lower part shows the variation with orbital angular momentum $l$ for the $n = 1$ value.
• Using $\hbar c = 197$ MeVfm and $mc^2 = 938$ MeV, we have $b^2 = 41.4 \text{ fm}^2 / (\hbar \omega)$

• Virial theorem applied to $U_0^{\text{HO}}(r) = \frac{1}{2} m \omega^2 r^2$,

$$\langle \phi_\alpha | T | \phi_\alpha \rangle = \langle \phi_\alpha | U | \phi_\alpha \rangle = \frac{1}{2} m \omega^2 \langle \phi_\alpha | r^2 | \phi_\alpha \rangle$$

together with

$$\langle \phi_\alpha | U | \phi_\alpha \rangle + \langle \phi_\alpha | T | \phi_\alpha \rangle = \left( N + \frac{3}{2} \right) \hbar \omega$$

gives the diagonal matrix elements of $r^2$:

$$\langle \phi_\alpha | r^2 | \phi_\alpha \rangle = \langle N | r^2 | N \rangle = (N + \frac{3}{2}) b^2 \quad (208)$$

and

$$\langle \phi_\alpha | T | \phi_\alpha \rangle = \left( N + \frac{3}{2} \right) \frac{\hbar \omega}{2} \quad (209)$$

• Maximum number of spin-1/2 fermions in state with given $N$ is $D_N = (N + 1)(N + 2)$, and number of states that can be occupied up to $N_{\text{max}}$ is

$$\sum_{N=0}^{N_{\text{max}}} D_N = \frac{1}{3} (N_{\text{max}} + 1)(N_{\text{max}} + 2)(N_{\text{max}} + 3) \quad (210)$$
The harmonic oscillator parameter $\omega$ reproduce the observed mean-square charge radius,

related to the mean-square radius for protons in the nucleus whose harmonic oscillator levels are filled up to $N_{\text{max}}$

$$\langle r^2 \rangle_p = \frac{\sum_{N=0}^{N_{\text{max}}} D_N \langle N| r^2 |N \rangle}{\sum_{N=0}^{N_{\text{max}}} D_N} = \frac{3}{4} (N_{\text{max}} + 2)b^2. \quad (211)$$

Compare to the observed proton mean-square radius as obtained from measured charge mean-square radius by

$$\langle r^2 \rangle_p = \langle r^2 \rangle_{ch} - \langle r^2 \rangle_o \quad (212)$$

where $\langle r^2 \rangle_o = 0.77 \text{ fm}^2$ is the mean-square charge radius of the proton.

For example, for the nucleus $^{16}$O, where the eight protons fill the $N = 0$ and $N = 1$ major shells, $N_{\text{max}} = 1$ and $\langle r^2 \rangle_p (^{16}$O) = $(9/4)b^2$.

The experimental root-mean-square (rms) charge radius for $^{16}$O is $\sqrt{\langle r^2 \rangle_{ch}} = 2.71 \text{ fm}$, and hence $\langle r^2 \rangle_p = 6.57 \text{ fm}^2$, $b^2 = 2.92 \text{ fm}^2$ and $\hbar \omega = 14.2$ MeV for the point protons.

Since there are equal numbers of neutrons and protons in $^{16}$O and since they should have about the same rms radii, it is a good approximation to use the same oscillator parameter for the neutrons as was obtained for protons.
**Figure 32:** Neutron single-particle states in $^{208}$Pb with three potential models. The numbers in square brackets are the maximum number of neutrons in that each level can contain, the following number is a running sum of the total. In addition the harmonic oscillator is labeled by the major quantum number $N = 2n + l$, the Woods Saxon is labeled by $n, l$ and the Woods-Saxon with spin-orbit is labeled by $n, l, 2j$. 

Harmonic Oscillator Potential  | Woods-Saxon Potential  | Woods-Saxon Plus Spin-Orbit Potential
• For heavy nuclei, use the large $N_{\text{max}}$ approximation for the sum, $Z \approx (N_{\text{max}} + 2)^3 / 3$, to obtain

$$\langle r^2 \rangle_p \approx \frac{3}{4}(3Z)^{1/3}b^2$$  \hspace{1cm} (213)

• For example, for $^{208}\text{Pb}$ ($Z = 82$) the experimental rms charge radius is 5.50 fm, giving $\langle r^2 \rangle_p = 29.4 \text{ fm}^2$, $b^2 = 6.27 \text{ fm}^2$ and $\hbar \omega_p = 6.6 \text{ MeV}$ for the point protons.

• Experimentally, the proton and neutron rms radii in $^{208}\text{Pb}$ are about equal $\langle r^2 \rangle_p \approx \langle r^2 \rangle_n$.

• The oscillator parameter for neutrons in $^{208}\text{Pb}$, obtained by using an rms radius of 5.5 fm and $N$ in place of $Z$, is $\hbar \omega_n = 7.6 \text{ MeV}$.

• Along the valley of stability the rms proton radii are approximately given by $1.08A^{1/3}$. Rewriting Eq. (213) in terms of $A \approx 2.5Z$, one obtains for heavy nuclei $\hbar \omega_p \approx 39A^{-1/3} \text{ MeV}$.

• A better smooth approximation for the proton oscillator parameter in both light and heavy nuclei is $\hbar \omega_p \approx 45A^{-1/3} - 25A^{-2/3} \text{ MeV}$
In both the square-well potential and in the harmonic oscillator potential 2, 8, 20 are the common shell-closures and these numbers correspond to the particularly stable nuclei $^4\text{He}$, $^{16}\text{O}$, $^{48}\text{Ca}$ and agree with the data. The harmonic oscillator potential has the larger degeneracy of the energy eigenvalues and the corresponding wavefunctions (the degeneracy on $2(n - 1) + l$ is split for the square-well potential).

Use of a more realistic potential does not help to solve the discrepancy!

Spin-orbit interaction needed:
It is clear that something fundamental is missing still. We shall pay attention to the origin of the spin-orbit force, coupling the orbital and intrinsic spin, that gives rise to a solution with the correct shell-closure numbers.
It was pointed out (independently by Mayer (1949,1950) and Haxel, Jensen, and Suess (1949,1950)) that a contribution to the average field, felt by each individual nucleon, should contain a spin-orbit term.

The corrected potential then becomes

\[ U(r) = -U_0 + \frac{1}{2} m \omega^2 r^2 - \frac{2}{\hbar^2} \alpha \hat{l} \cdot \hat{s}. \]  

(214)

The spin-orbit term, with the scalar product of the orbital angular momentum operator \( \hat{l} \) and the intrinsic spin operator \( \hat{s} \), can be rewritten using the total nucleon angular momentum operator \( \hat{j} \), as

\[ \hat{j} = \hat{l} + \hat{s}, \]  

(215)

and

\[ \hat{l} \cdot \hat{s} = \frac{1}{2} \left( \hat{j}^2 - \hat{l}^2 - \hat{s}^2 \right). \]  

(216)

The operators \( \hat{j}^2, \hat{s}^2, \) and \( \hat{j}^2 \) form a set of commuting angular momentum operators, so they have a set of common wavefunctions.
These wavefunctions, characterized by the quantum numbers corresponding to the four commuting operators $\hat{J}^2$, $\hat{S}^2$, $\hat{j}^2$, and $\hat{j}_z$ result from angular momentum coupling the orbital ($Y^m_l(\theta, \varphi)$) and spin ($\chi^{m_s}_{1/2}(\sigma)$) wavefunctions. We denote these as

$$\phi(\vec{r}, n(l_{1/2})jm) \equiv R_{nl}(r) \sum_{m_l, m_s} \langle lm_l, \frac{1}{2} m_s | jm \rangle Y^m_l(\theta, \varphi) \chi^{m_s}_{1/2}(\sigma),$$

(217)

(with $m = m_l + m_s$). These wavefunctions correspond to good values of $(l_{1/2})jm$ and we derive the eigenvalue equation

$$\hat{J} \cdot \hat{S} \phi(\vec{r}, n(l_{1/2})jm) = \frac{\hbar^2}{2} \left[ j(j+1) - l(l+1) - \frac{3}{4} \right] \phi(\vec{r}, n(l_{1/2})jm)$$

(218)

Since we have the two orientations

$$j = l \pm \frac{1}{2}$$

(219)

we obtain

$$\hat{J} \cdot \hat{S} \phi(\vec{r}, n(l_{1/2})jm) = \frac{\hbar^2}{2} \left\{ \begin{array}{ll} l, & j = l + 1/2 \\ -(l+1), & j = l - 1/2 \end{array} \right\} \phi(\vec{r}, n(l_{1/2})jm).$$

(220)
• The *effective* potential then becomes slightly different for the two orientations i.e.

\[ U(r) = -U_0 + \frac{1}{2} m\omega^2 r^2 + \alpha \left\{ \begin{array}{c} -l \\ + (l + 1) \end{array} \right\}, \quad j = \left\{ \begin{array}{c} l + \frac{1}{2} \\ l - \frac{1}{2} \end{array} \right\} \ (221) \]

• The potential is more attractive for the parallel \( j = l + 1/2 \) orientation, relative to the anti-parallel situation. The corresponding energy eigenvalues now become

\[ \varepsilon_{n(l + \frac{1}{2})} = \hbar \omega \left[ 2(n - 1) + l + \frac{3}{2} \right] - U_0 + \alpha \left\{ \begin{array}{c} -l \\ l + 1 \end{array} \right\}, \quad j = \left\{ \begin{array}{c} l + \frac{1}{2} \\ l - \frac{1}{2} \end{array} \right\} \ (222) \]

and the degeneracy in the \( j = l \pm \frac{1}{2} \) coupling is broken. The final single-particle spectrum now becomes as given in figure.
Figure 33: Single-particle spectrum up to $N = 6$. The various contributions to the full orbital and spin-orbit splitting are presented. Partial and accumulated nucleon numbers are drawn at the extreme right. (Taken from Mayer and Jensen, Elementary Theory of Nuclear Shell Structure, ©1955 John Wiley & Sons. Reprinted by permission.)
Woods-Saxon potential

- A convenient *phenomenological* choice for the one-body potential, for modelling properties of bound-state and continuum single-particle wavefunctions.
- cannot be used for the total binding energy since it is not based upon a specific two-body interaction
- Parameters from best fit of nuclear single-particle energies and nuclear radii \((O = \text{orbital}, \ SO = \text{spin-orbit})\):

  \[
  U(r) = U_O(r) + U_{SO}(r)\hat{l} \cdot \hat{s} + U_C(r), \quad \text{where}
  \]

  \[
  U_O(r) = U_O f_O(r) = \frac{U_O}{1 + \exp[(r - R_O)/a_O]}, \quad \text{spin-independent central potential of Fermi shape}
  \]

  \[
  U_{SO}(r) = U_{SO} \frac{1}{r} \frac{d}{dr} f_{SO}(r), \quad \text{the spin-orbit potential with}
  \]

  \[
  f_{SO}(r) = \frac{1}{1 + \exp[(r - R_{SO})/a_{SO}]},
  \]

and \(U_C(r)\) is the Coulomb potential for protons:
\[ U_C(r) = \frac{Ze^2}{4\pi\epsilon_0} \left\{ \begin{array}{ll}
\frac{1}{r} & \text{for } r \geq R_C \\
\frac{1}{R_C} \left[ \frac{3}{2} - \frac{r^2}{2R_C^2} \right] & \text{for } r \leq R_C
\end{array} \right. \]

- The radial parameters \( R_O, R_{SO}, \) and \( R_C \) are usually expressed as:
  \[ R_i = r_i A^{1/3}. \]

- Average proton-neutron potential is stronger than the average neutron-neutron (or proton-proton) potential:
  \[ U_{O,p} = U_0 + \frac{N - Z}{A} U_1 \quad \text{for protons} \]
  \[ U_{O,n} = U_0 - \frac{N - Z}{A} U_1 \quad \text{for neutrons} \]

- Typical values: \( U_0 = -53 \text{ MeV}, U_1 = -30 \text{ MeV} \) and \( U_{SO} = 22 \text{ MeV} \), for the strengths, and \( r_O = r_{SO} = 1.25 \text{ fm} \) and \( a_O = a_{SO} = 0.65 \text{ fm} \) for the geometry. For the Coulomb term the radius is a little smaller with \( r_C = 1.20 \text{ fm} \).\(^6\)

\(^6\)One can find in the literature many other sets of parameters which are better for specific nuclei or mass regions.
**Figure 34**: Woods-Saxon potential for a 1d_{5/2} proton or neutron with a $^{16}$O (left) or $^{208}$Pb (right) core, together with the total potential and the contributing potentials as described in the text.
Woods-Saxon neutron single-particle energies for nuclei near the valley of stability as a function of $A$. With increasing $A$, more single-particle states become bound but the energy of the most loosely bound filled orbit is always around 8 MeV (Bohr & Mottelson 1969).
• The extreme single-particle shell model applies to some rather simple situations, where a single *un-paired* nucleon (or hole) is outside a *closed core*. A more realistic description is needed.

• Many properties are influenced by pairing correlations:
  • Gap in the excitation spectra of even-$A$ nuclei
  • Odd-even mass differences
  • Rotational moments of inertia and particle alignments, etc. . .