5 Thermal history of the early universe

5.1 Timescale of the early universe

We will now apply the thermodynamics discussed in the previous section to the evolution of the early universe. It is useful to keep in mind some simple relations between time, distance and temperature in a radiation-dominated universe. Spatial curvature can be neglected in the early universe, so the metric is

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right) . \tag{5.1}$$

and the Friedmann equation is

$$3H^2 = 8\pi G_\mathrm{N}\rho(T) = \frac{\pi^2}{30} g_*(T) \frac{T^4}{M_\mathrm{Pl}^2} , \tag{5.2}$$

where we have written Newton's constant in terms of the Planck mass, $M_\mathrm{Pl} \equiv 1/\sqrt{8\pi G_\mathrm{N}} \approx 2.436 \times 10^{21}$ MeV. To integrate this equation exactly we would need to calculate numerically the function $g_*(T)$, taking into account all the annihilations. For most of the time, however, $g_*(T)$ changes slowly, so we can approximate $g_*(T) =$ const. Then $T \propto a^{-1}$ and $H \propto a^{-2}$, so we get the following relation between the age of the universe t and the Hubble parameter H:

$$t = \frac{1}{2} H^{-1} = \sqrt{\frac{45}{2\pi^2}} \frac{1}{\sqrt{g_*}} \frac{M_\mathrm{Pl}}{T^2} \approx \frac{1.51}{\sqrt{g_*}} \frac{M_\mathrm{Pl}}{T^2} \approx \frac{2.42}{\sqrt{g_*}} \left(\frac{\mathrm{T}}{\mathrm{MeV}}\right)^{-2} \mathrm{s} . \tag{5.3}$$

We thus have

$$a \propto T^{-1} \propto t^{1/2} .$$

This approximate result (5.3) will be sufficient for us as far as the time scale is concerned[1], but for the relation between a and T, we need to use the more exact result derived in section 4.5.

The distance to the horizon (i.e. proper comoving distance to $t = 0$, or $z = \infty$) is

$$d_\mathrm{hor}(t) = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')} = 2t = H^{-1} . \tag{5.4}$$

In the radiation-dominated early universe, the distance to the horizon is equal to the Hubble length, so we can use the terms "horizon length", "horizon" and "Hubble length" interchangeably. This is often also done for other eras, when the two are not equal. In particular, when a period of inflation is added to early times, the particle horizon will be much larger than the Hubble length at late times – we will come to this the second part of the course.

5.2 Particle content

The primordial soup initially consists of all the different species of elementary particles. Their masses range from the heaviest known elementary particle, the top quark ($m = 173$ GeV) down to the lightest particles, the electron ($m = 511$ keV), neutrinos

[1] Usually the error from ignoring the time-dependence of $g_*(T)$ is negligible, since the time scales of earlier events are so much shorter.

($m \lesssim 2$ eV) and the photon ($m = 0$). In addition to the particles of the Standard Model, there are presumably other, so far undiscovered, species. In particular, we will discuss dark matter particles in chapter 7. As the temperature falls, the various particle species become nonrelativistic and annihilate at different times.

The particles of the Standard Model are listed in table 1. The internal degrees of freedom for quarks are 2 for spin, 2 for having both left- and right-handed components (this is a better parametrisation than counting particles and antiparticles) and 3 for colour. Electrons, muons and and taus don't have colour, but otherwise the counting is the same. In the Standard Model, there are only left-handed neutrinos, so the only have the spin degeneracy factor. Massless spin 1 particles like the photon only have 2 spin degrees of freedom, while massive ones like $W^{\pm}$ and Z have three (note that W^+ and W^- are counted separately).

The effective number of degrees of freedom $g_*(T)$ (solid), $g_{*p}(T)$ (dashed) and $g_{*s}(T)$ are plotted in figure 1 as a function of temperature. In table 2 we list some important events in the early universe.

Table 1: The particles in the Standard Model
Particle Data Group, 2014 [2]

Quarks	t	173.2 ± 0.9 GeV	$\bar{t}$	spin $\frac{1}{2}$	$g = 2 \cdot 2 \cdot 3 = 12$	
	b	4.18 ± 0.03 GeV	$\bar{b}$	3 colours		
	c	1.275 ± 0.025 GeV	$\bar{c}$			
	s	95 ± 5 MeV	$\bar{s}$			
	d	4.5–5.3 MeV	$\bar{d}$			
	u	1.8–3.0 MeV	$\bar{u}$			
						72
Gluons	8 massless bosons			spin 1	$g = 2$	16
Leptons	τ^-	1776.82 ± 0.16MeV	τ^+	spin $\frac{1}{2}$	$g = 2 \cdot 2 = 4$	
	μ^-	105.658 MeV	μ^+			
	e^-	510.999 keV	e^+			
						12
	ν_τ	< 2 eV	$\bar{\nu}_\tau$	spin $\frac{1}{2}$	$g = 2$	
	ν_μ	< 2 eV	$\bar{\nu}_\mu$			
	ν_e	< 2 eV	$\bar{\nu}_e$			
						6
Electroweak gauge bosons	$W^{\pm}$	80.385 ± 0.015 GeV		spin 1	$g = 3$	
	Z^0	91.1876 ± 0.0021 GeV				
	γ	0 ($< 1 \times 10^{-18}$ eV)			$g = 2$	
						11
Higgs boson	H	125.7 ± 0.4 GeV		spin 0	$g = 1$	1

$$g_f = 72 + 12 + 6 = 90$$
$$g_b = 16 + 11 + 1 = 28$$

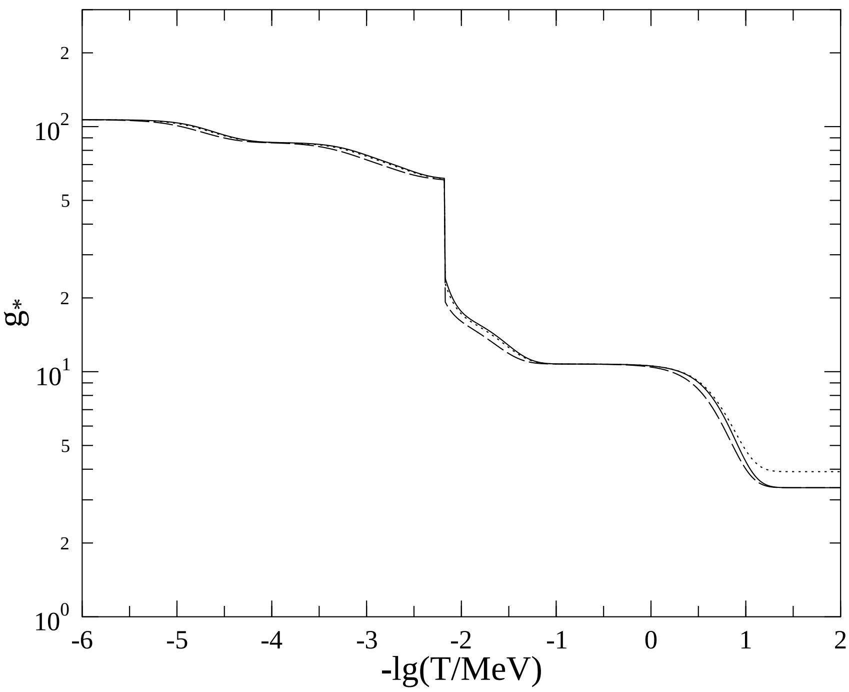

Figure 1: The functions $g_*(T)$ (solid), $g_{*p}(T)$ (dashed), and $g_{*s}(T)$ (dotted) for Standard Model particle content.

For $T > m_t = 173$ GeV, all known particles are relativistic. Adding up their internal degrees of freedom we get

$$
\begin{aligned}
g_b &= 28 && \text{gluons } 8\times2\text{, photons 2, } W^\pm \text{ and } Z^0\ 3\times3\text{, and Higgs 1} \\
g_f &= 90 && \text{quarks } 12\times6\text{, charged leptons } 6\times2\text{, neutrinos } 3\times2 \\
g_* &= 106.75\ .
\end{aligned}
$$

The electroweak (EW) crossover takes place at the temperature 160 GeV [1]. Sometimes this process is called the electroweak phase transition. However, in the Standard Model, it is a smooth crossover from one regime to another, and thermodynamic quantities remain continuous. In some extensions of the Standard Model, there is a phase transition, where the system is not in thermal equilibrium. This may have important cosmological consequences (in particular, it may determine the baryon-antibaryon asymmetry observed in the universe), depending on the way the electroweak phase transition happens. We will not discuss details of the electroweak crossover, for our purposes it is enough to know that g_* is the same before and after the transition, at least in the Standard Model. Going to earlier times and higher temperatures, we expect g_* to get larger than 106.75 as new physics and thus far unknown particle species comes to play.

Let us now follow the history of the universe starting at the time when the EW crossover has already happened. We have $T \sim 160$ GeV, $t \sim 10$ ps, and t quark annihilation is ongoing. (Recall that the transition from relativistic to non-relativistic behaviour is not complete until about $T \approx m/6 \approx 30$ GeV.) The Higgs boson annihilates next, and then the gauge bosons $W^\pm$ and Z^0. At $T \sim 10$ GeV,

Electroweak crossover	$T \sim 160$ GeV	$t \sim 10$ ps
QCD crossover	$T \sim 100$ MeV	$t \sim 30\mu$s
Neutrino decoupling	$T \sim 1$ MeV	$t \sim 1$ s
Electron-positron annihilation	$T < m_e = 0.5$ MeV	$t \sim 10$ s
Big Bang Nucleosynthesis	$T \sim 50$–100 keV	$t \sim 3$–30 min
Matter-radiation equality	$T \sim 0.8$ eV $= 9000$ K	$t \sim 50\,000$ yr
Recombination + photon decoupling	$T \sim 0.3$ eV $= 3000$ K	$t \sim 380\,000$ yr

Table 2: Early universe events.

we have $g_* = 86.25$. Next the b and c quarks annihilate, followed by the τ lepton. If the s quark would also have had time to annihilate, we would reach $g_* = 51.25$.

5.3 QCD crossover

In the middle of the s quark annihilation, something else happens, however: matter undergoes the *QCD crossover* (also called the quark–hadron crossover). This takes place at $T \sim 100$ MeV, $t \sim 30$ μs. The colour forces between quarks and gluons become important, so the formulae for the energy density for free particles in chapter 4 no longer apply. The quarks and gluons form bound three-quark systems, called *baryons*, and quark-antiquark pairs, called *mesons*. (Together, these bound states of quarks are known as *hadrons*.) Baryons are fermions, mesons are bosons. After that, there are no more free quarks and gluons; the *quark-gluon plasma* has become a *hadron plasma*. The lightest baryons are the nucleons: the proton and the neutron. The lightest mesons are the pions: $\pi^\pm$, π^0.

There are many different species of baryons and mesons, but all except pions are nonrelativistic below the QCD crossover temperature. Thus the only Standard Model particle species left in large numbers are pions, muons, electrons, neutrinos and photons. For pions, $g = 3$, so we have $g_* = 17.25$.

Table 3: History of $g_*(T)$

$T \sim 200$ GeV	all present	106.75	
$T \sim 160$ GeV	EW crossover	(no effect)	
$T < 170$ GeV	top annihilation	96.25	
$T < 120$ GeV	H^0	95.25	
$T < 80$ GeV	$W^\pm$, Z^0	86.25	
$T < 4$ GeV	bottom	75.75	
$T < 1$ GeV	charm, τ^-	61.75	
$T \sim 100$ MeV	QCD crossover	17.25	(u,d,g$\to \pi^{\pm,0}$, $37 \to 3$)
$T < 100$ MeV	$\pi^\pm$, π^0, μ^-	10.75	$e^\pm$, ν, $\bar{\nu}$, γ left
$T < 500$ keV	e^- annihilation	(7.25)	$2 + 5.25(4/11)^{4/3} = 3.36$

The above table gives the value $g_*(T)$ would have after the annihilation is over, assuming the next annihilation would not have begun yet. In reality the annihilations overlap in many cases. The temperature value on the left is (apart from the crossover temperatures) the approximate mass of the particle in question and indicates roughly when the annihilation begins. The temperature is much smaller when the annihilation ends. The top quark receives its mass in the EW crossover, so its annihilation does not begin before the crossover.

5.4 Neutrino decoupling and electron-positron annihilation

Soon after the QCD crossover, pions and muons annihilate and for $T = 20$ MeV $\to 1$ MeV, we have $g_* = 10.75$. Next the electrons annihilate, but to discuss the e^+e^- annihilation we need a bit more details.

So far we have assumed that all particle species have the same temperature, i.e. particle interactions keep them in thermal equilibrium. Neutrinos, however, feel only the weak interaction. The weak interaction is actually not that weak when particle energies are close to (or higher than) the masses of the W and Z bosons, which mediate the interaction. But as the temperature, and thus mean energy of particles, falls, the weak interaction becomes rapidly weaker.

A particle species falls out of chemical equilibrium when interactions become too weak to maintain it in touch with the other species as the universe expands. This happens when the *interaction rate* Γ becomes smaller than the expansion rate, $\Gamma < H$. The interaction rate Γ has units of 1/time, and it can be interpreted as the frequency of particle interactions. The limit $\Gamma < H$ can roughly be understood as saying that if particles on average have less than one interaction per Hubble time, the distribution cannot keep up with the expansion. The interaction rate can be written as $\Gamma = n\langle\sigma|v|\rangle$, where n is number density of the particles, σ is *interaction cross section*, v is particle velocity and the brackets are average over the phase space. If the cross section is independent of velocity, we can take it out of the average. If the particles are ultrarelativistic, we can approximate $|v| = 1$, in which case we have simply $\Gamma = n\sigma$. The cross section has units of area, and it expresses the strength of the interaction[2].

For the weak interaction processes relevant for neutrinos, the cross section is $\sigma \sim G_F^2 T^2$, where $G_F \approx 1.17 \times 10^{-5}$ GeV^{-2} is the Fermi constant. The interaction rate is then $\Gamma = n\sigma v \sim G_F^2 T^5$, where n is the number density and $v \approx 1$ is typical neutrino velocity. According to the Friedmann equation, $H \sim \sqrt{\rho/M_{\rm Pl}^2} \sim T^2/M_{\rm Pl}$. So we have $\Gamma/H \sim G_F^2 M_{\rm Pl} T^3 \sim (T/\text{ MeV})^3$. So, neutrinos *decouple* close to $T \sim 1$ MeV, after which they move practically freely, without interactions.

Even though neutrinos are no longer in chemical equilibrium, they remain in thermal equilibrium as long as the temperature of the particle soup also evolves like $T \propto a^{-1}$, so that $T_\nu = T$. However, annihilations will cause a deviation from $T \propto a^{-1}$. The next annihilation event is the electron-positron annihilation.

As the number of relativistic degrees of freedom is reduced, energy density and entropy are transferred from electrons and positrons to photons, but not to neutrinos, in the annihilation reactions

$$e^+ + e^- \quad \to \quad \gamma + \gamma \ .$$

The photons are thus heated relative to neutrinos (the photon temperature does not

[2]This terminology comes from particle physics. The idea is that if you consider a beam of classical particles randomly directed at a target with total area A, and classical particles take up an are σ of it, the probability of crossing a particle and hence interacting is σ/A.

fall as much). In the electron-positron annihilation, g_{*s} changes from

$$g_{*s} = g_* = \underset{\gamma}{2} + \underset{e^\pm}{3.5} + \underset{\nu}{5.25} = 10.75 \tag{5.5}$$

to

$$g_{*s} = 2 + 5.25 \left(\frac{T_\nu}{T}\right)^3 . \tag{5.6}$$

For time 1 before the annihilation and time 2 after it, we have from (4.34)

$$2a_2^3T_2^3 + 5.25a_2^3T_{\nu 2}^3 = 10.75a_1^3T_1^3 . \tag{5.7}$$

Before the electron-positron annihilation, the neutrino temperature was the same as the temperature of the other species, so $a_1^3T_1^3 = a_1^3T_{\nu 1}^3 = a_2^3T_{\nu 2}^3$, where we have used the fact that $T_\nu \propto a^{-1}$ throughout, since neutrinos are relativistic and they are not heated by the electron-positron annihilation. We thus have from (5.7)

$$10.75 = 2\left(\frac{T}{T_\nu}\right)^3 + 5.25 ,$$

from which we solve the neutrino temperature after e^+e^--annihilation[3],

$$\begin{aligned} T_\nu &= \left(\frac{4}{11}\right)^{\frac{1}{3}} T = 0.714\,T \\ g_{*s}(T) &= 2 + 5.25 \cdot \frac{4}{11} = 3.909 \\ g_*(T) &= 2 + 5.25\left(\frac{4}{11}\right)^{\frac{4}{3}} = 3.363. \end{aligned} \tag{5.8}$$

These relations remain true for the photon+neutrino background as long as the neutrinos stay ultrarelativistic ($m_\nu \ll T$). The neutrinos are no longer in chemical or thermal equilibrium, but they are still in kinetic equilibrium, i.e. their distribution function has the thermal shape.

If the neutrinos masses were so small that they could be ignored, the above relation would apply even today, when the photon (the CMB) temperature is $T = T_0 = 2.725$ K $= 0.2348$ meV, giving the neutrino background temperature $T_{\nu 0} = 0.714 \cdot 2.725$ K $= 1.945$ K $= 0.1676$ meV. However, *neutrino oscillation* experiments in the 1990s established that neutrinos have masses which are at least in the meV range[4], and there is an upper limit of about 2 eV from direct detection experiments and cosmology. Therefore, the neutrino background is non-relativistic today. As neutrinos become non-relativistic, they fall out of kinetic equilibrium, because the shape of the thermal distribution function is not preserved as the momenta redshift to the value $p \sim m$. Once neutrinos become very non-relativistic, with typical values of the momenta $p \ll m$, the distribution function again has the thermal shape, but with a different temperature scaling.

[3]To be more precise, neutrino decoupling was not complete when e^+e^--annihilation began, so some of the energy and entropy did leak to the neutrinos. Therefore the neutrino energy density after e^+e^--annihilation is about 1.3% higher (at a given T) than the above calculation gives. The neutrino distribution also deviates slightly from kinetic equilibrium.

[4]Specifically, the oscillations show that the mass differences between the neutrinos are of the order meV. In principle, the lightest neutrino could be massless.

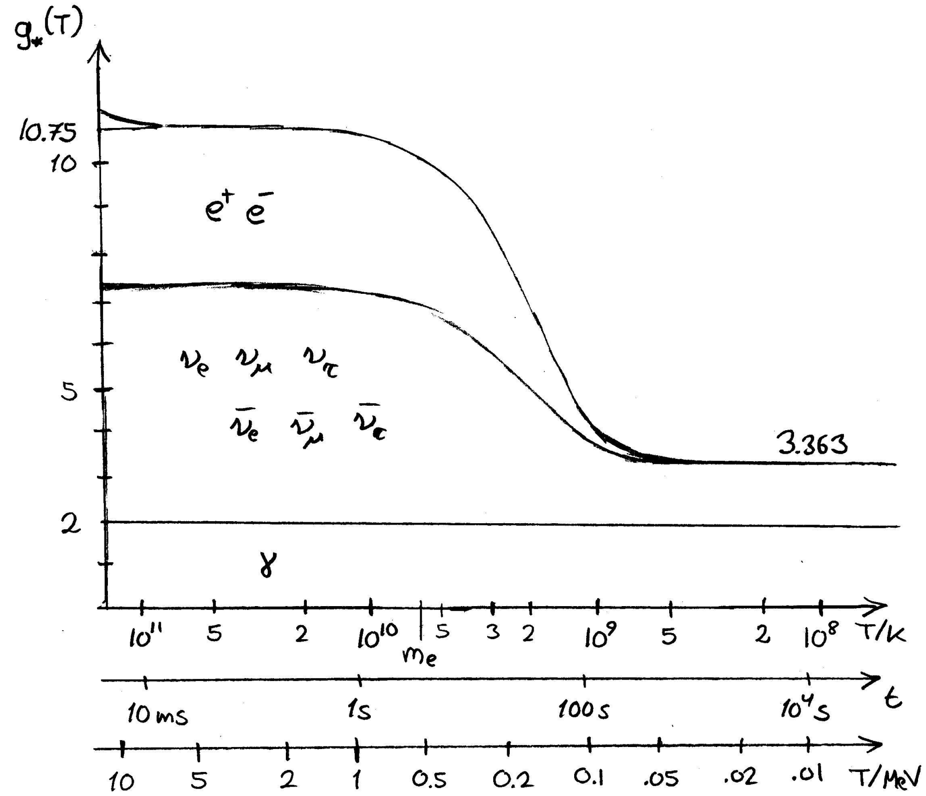

Figure 2: The evolution of the energy density, or rather, $g_*(T)$, and its different components through electron-positron annihilation. Since $g_*(T)$ is defined as $\rho/(\pi^2 T^4/30)$, where T is the photon temperature, the photon contribution appears constant. If we had plotted $\rho/(\pi^2 T_\nu^4/30) \propto \rho a^4$ instead, the neutrino contribution would appear constant, and the photon contribution would increase at the cost of the electron-positron contribution, which would better reflect what is going on.

5.5 Matter

We noted that the early universe is dominated by the relativistic particles, and we can forget the nonrelativistic particles when we are considering the dynamics of the universe. We followed one species after another becoming nonrelativistic and disappearing from the picture, until only photons (the cosmic background radiation) and neutrinos were left, and the neutrinos had stopped interacting.

We now return to look in more detail what happens to nucleons and electrons. We found that they annihilated with their antiparticles when the temperature fell below their respective rest masses. For nucleons, the annihilation began immediately after they were formed in the QCD crossover. There were however slightly more particles than antiparticles, and this small excess of particles was left over. (This has to be the case because we observe electrons and nucleons today – we'll be more quantitative in chapter 7.) This means that the chemical potential μ_B associated with baryon number differs from zero (it is positive). Baryon number is a conserved quantity in the eras we are considering (though not before the electroweak crossover). Baryon number resides today in nucleons (protons and neutrons; since the proton is lighter than the neutron, free neutrons have decayed into protons, but there are neutrons in atomic nuclei) because they are the lightest baryons. The universe is electrically neutral, and the negative charge lies in the electrons, the lightest particles with negative charge. Therefore the number of electrons equals the number of protons.

We get the number densities etc. of the electrons and the nucleons from the equations of chapter 4. But what is the value of the chemical potential μ? For each species, we get $\mu(T)$ from the conserved quantities[5]. The baryon number resides in the nucleons,

$$n_\mathrm{B} \;=\; n_\mathrm{N} - n_{\bar{\mathrm{N}}} \;=\; n_\mathrm{p} + n_\mathrm{n} - n_{\bar{\mathrm{p}}} - n_{\bar{\mathrm{n}}}\,. \tag{5.9}$$

Let us define the parameter η, the baryon-photon ratio today,

$$\eta \equiv \frac{n_\mathrm{B}(t_0)}{n_\gamma(t_0)}\,. \tag{5.10}$$

From observations we know that $\eta \approx 6 \times 10^{-10}$. (We will take a closer look at the observational value in the next chapter.) Since baryon number is conserved, $n_\mathrm{B} V \propto n_\mathrm{B} a^3$ stays constant, so

$$n_\mathrm{B} \propto a^{-3}\,. \tag{5.11}$$

After electron-positron annihilation, we have $n_\gamma \propto a^{-3}$, so we get

$$n_\mathrm{B}(T) = \eta n_\gamma = \eta \frac{2\zeta(3)}{\pi^2} T^3 \qquad \text{for } T \ll m_\mathrm{e}\,. \tag{5.12}$$

We can put (5.11) and (5.12) together and replace a^{-3} using the relation (4.34) between the temperature and the scale factor to obtain

$$n_\mathrm{B}(T) = \eta \frac{2\zeta(3)}{\pi^2} \frac{g_{*s}(T)}{g_{*s}(T_0)} T^3\,. \tag{5.13}$$

[5] In general, the recipe to find how the thermodynamical parameters evolve in the expanding FRW universe is to use the conservation laws of the conserved number densities, entropy conservation and the energy continuity equation to find how the number densities and energy densities evolve. The other thermodynamical parameters then evolve so as to satisfy these requirements.

For $T < 10$ MeV we have

$$n_{\bar{\mathrm{N}}} \ll n_{\mathrm{N}} \qquad \text{and} \qquad n_{\mathrm{N}} \equiv n_{\mathrm{n}} + n_{\mathrm{p}} = n_{\mathrm{B}} \,.$$

In the next chapter, we will discuss big bang nucleosynthesis, i.e. how the protons and neutrons form atomic nuclei. Approximately one quarter of all nucleons (all neutrons and roughly the same number of protons) form nuclei ($A > 1$) and three quarters remain as free protons. Let us denote by n_{p}^* and n_{n}^* the total number densities of protons and neutrons including those in nuclei (and also those in atoms), whereas we shall use n_{p} and n_{n} for the number densities of *free* protons and neutrons, which are not bound to each other or electrons. We thus write

$$n_{\mathrm{N}}^* \equiv n_{\mathrm{n}}^* + n_{\mathrm{p}}^* = n_{\mathrm{B}} \,.$$

In the same manner, for $T < 10$ keV we have

$$n_{\mathrm{e}^+} \ll n_{\mathrm{e}^-} \qquad \text{and} \qquad n_{\mathrm{e}^-} = n_{\mathrm{p}}^* \,.$$

At this time ($T \sim 10$ keV $\to 1$ eV) the universe contains a relativistic photon and neutrino background ("radiation") and nonrelativistic free electrons, protons, and nuclei ("matter"). Since $\rho \propto a^{-4}$ for radiation, but $\rho \propto a^{-3}$ for matter, the energy density in radiation falls eventually below the energy density in matter—the universe becomes *matter-dominated*.

The above discussion is in terms of the known particle species. Today there is much indirect observational evidence for the existence of what is called *cold dark matter* (CDM), which presumably consists of some yet undiscovered species of particles. The CDM particles interact weakly with normal matter (they decouple early), and their energy density contribution should be small deep in the radiation-dominated era, so they do not affect the above discussion much. They become nonrelativistic early and dominate the matter density of the universe today (there is about five to six times as much mass in CDM as there is in baryons). Thus CDM causes the universe to become matter-dominated earlier than if the matter consisted of nucleons and electrons only. The CDM will be important later when we discuss the formation of structures in the universe. The time of matter-radiation equality t_{eq} is calculated in an exercise at the end of this chapter.

5.6 Recombination

Radiation (photons) and matter (electrons, protons, and nuclei) remained in thermal equilibrium for as long as there were lots of free electrons. When the temperature became low enough the electrons and nuclei combined to form neutral atoms, an event known as *recombination*[6], and the density of free electrons fell sharply. The *photon mean free path* grew rapidly and became longer than the horizon distance. Thus the universe became *transparent*. Photons and matter *decoupled*, i.e. their interaction was no longer able to maintain them in thermal equilibrium with each other. After this, by T we refer to the photon temperature. Today, these photons are the CMB, and $T = T_0 = 2.725$ K. (After photon decoupling, the matter temperature

[6]This is the first time when nuclei and electrons combine, so the term *re*combination, adopted from chemistry, is somewhat of a misnomer.

fell at first faster than the photon temperature, but structure formation then heated up the matter to different temperatures in different places.)

The relevant interaction here is not weak interaction, as in the case of the neutrinos, but instead electromagnetic interaction between photons and electrons. The interaction rate is $\Gamma \sim n_e \sigma_T$, where $\sigma_T = \frac{8\pi}{3}\alpha^2/m_e^2 \approx 2 \times 10^{-3}$ MeV^{-2} is the Thomson cross-section, and $\alpha \approx 1/137$ is the electromagnetic coupling constant. (The $1/m^2$ factor shows that interactions between photons and nuclei are not important, as they are suppressed by the large masses of the nuclei.) Finding the photon decoupling era is a bit more involved than in the neutrino case, as the evolution of the electron number density is more complicated.

To simplify the discussion, let us ignore other nuclei than protons (over 90%, by number, of the nuclei are protons, and almost all the rest are ^{4}He nuclei). Let us denote the number density of *free* protons by n_p, free electrons by n_e, and hydrogen atoms by n_H. Since the universe is electrically neutral, $n_\mathrm{p} = n_\mathrm{e}$. The conservation of baryon number gives $n_\mathrm{B} = n_\mathrm{p} + n_\mathrm{H}$. From chapter 4 we have

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{\frac{\mu_i - m_i}{T}}. \tag{5.14}$$

For as long as the reaction

$$p + e^- \leftrightarrow \mathrm{H} + \gamma \tag{5.15}$$

is in chemical equilibrium the chemical potentials are related by $\mu_\mathrm{p} + \mu_\mathrm{e} = \mu_\mathrm{H}$ (since $\mu_\gamma = 0$). Using this we get the relation

$$n_\mathrm{H} = \frac{g_\mathrm{H}}{g_\mathrm{p} g_\mathrm{e}} n_\mathrm{p} n_\mathrm{e} \left(\frac{m_\mathrm{e} T}{2\pi}\right)^{-3/2} e^{B/T}, \tag{5.16}$$

between the number densities. Here $B = m_\mathrm{p} + m_\mathrm{e} - m_\mathrm{H} = 13.6$ eV is the *binding energy* of hydrogen. The numbers of internal degrees of freedom are $g_\mathrm{p} = g_\mathrm{e} = 2$, $g_\mathrm{H} = 4$. Outside the exponent we approximated $m_\mathrm{H} \approx m_\mathrm{p}$. Defining the *fractional ionisation*

$$x \equiv \frac{n_\mathrm{p}}{n_\mathrm{B}} , \tag{5.17}$$

equation (5.16) becomes

$$\frac{1-x}{x^2} = \frac{4\sqrt{2}\,\zeta(3)}{\sqrt{\pi}} \eta \left(\frac{T}{m_\mathrm{e}}\right)^{3/2} e^{B/T}, \tag{5.18}$$

the *Saha equation* for ionisation in thermal equilibrium. When $B \ll T \ll m_\mathrm{e}$, the RHS $\ll 1$, so $x \sim 1$, and almost all protons and electrons are free. As temperature falls, $e^{B/T}$ grows, but since both η and $(T/m_\mathrm{e})^{3/2}$ are $\ll 1$, the temperature needs to fall to $T \ll B$, before the whole expression becomes large (~ 1 or $\gg 1$).

The ionisation fraction at first follows the equilibrium result of (5.18) closely, but as this equilibrium fraction begins to fall rapidly, the true ionisation fraction begins to lag behind. As the number densities of free electrons and protons fall, it becomes more difficult for them to find each other to "recombine", and they are no longer able to maintain chemical equilibrium for the reaction (5.15). To find the correct ionisation evolution, $x(t)$, requires then a more complicated calculation involving the reaction cross section of this reaction. See figures 3 and 4.

Although the equilibrium formula is thus not enough to give us the true ionisation evolution, its benefit is twofold:

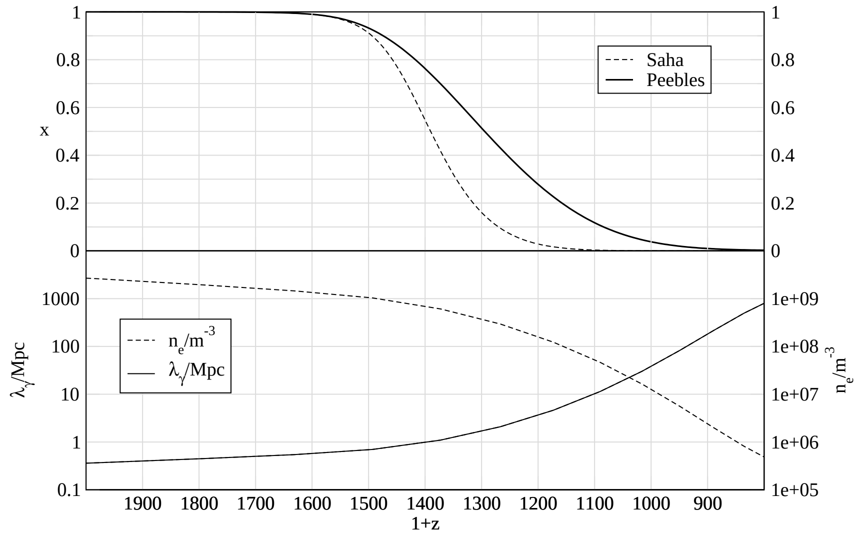

Figure 3: Recombination. In the top panel the dashed curve gives the equilibrium ionisation fraction as given by the Saha equation. The solid curve is the true ionisation fraction, calculated using the actual reaction rates (original calculation by Peebles). You can see that the equilibrium fraction is followed at first, but then the true fraction lags behind. The bottom panel shows the free electron number density n_e and the photon mean free path λ_γ. The latter is given in comoving units, i.e., the distance is scaled to the corresponding present distance. This figure is for $\eta = 8.22 \times 10^{-10}$. (Figure by R. Keskitalo.)

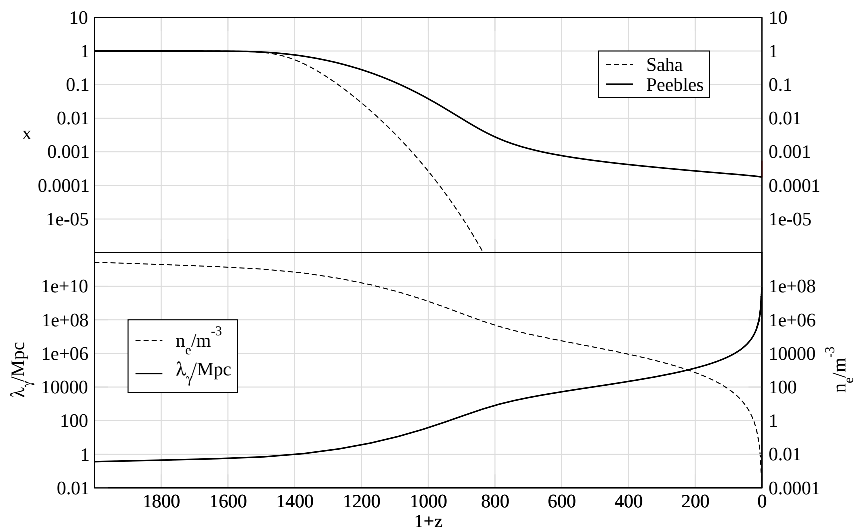

Figure 4: Same as figure 3, but with a logarithmic scale for the ionisation fraction, and the redshift scale extended to present time ($z = 0$ or $1 + z = 1$). You can see that a residual ionisation $x \sim 10^{-4}$ remains. This figure does not include reionisation, which happened around $z \sim 10$. (Figure by R. Keskitalo.)

1. It tells us when recombination begins. While the equilibrium ionisation changes only very slowly, it is easy to stay in equilibrium. Thus things won't start to happen until the equilibrium fraction begins to change a lot.

2. It gives the initial conditions for the more complicated calculation that will give the true evolution.

A similar situation holds for many other events in the early universe, e.g. big bang nucleosynthesis.

Recombination is not instantaneous. Let us define the recombination temperature $T_{\rm rec}$ as the temperature where $x = 0.5$. Now $T_{\rm rec} = T_0(1+z_{\rm rec})$ since $1+z = a^{-1}$ and the photon temperature falls as $T \propto a^{-1}$. (Since $\eta \ll 1$, the energy release in recombination is negligible compared to ρ_γ; and after photon decoupling photons travel freely maintaining kinetic equilibrium with $T \propto a^{-1}$.)

We get (for $\eta \sim 10^{-9}$)

$$\begin{aligned} T_{\rm rec} &\sim 0.3 \text{ eV} \\ z_{\rm rec} &\sim 1300. \end{aligned}$$

You might have expected that $T_{\rm rec} \sim B$. Instead we found $T_{\rm rec} \ll B$. The main reason for this is that $\eta \ll 1$. This means that there are very many photons for each hydrogen atom. Even when $T \ll B$, the high-energy tail of the photon distribution contains photons with energy $E > B$ so that they can ionise a hydrogen atom.

The photon decoupling takes place somewhat later, at $T_{\rm dec} \equiv (1 + z_{\rm dec})T_0$, when the ionisation fraction has fallen enough. We define the photon decoupling time to be the time when the photon mean free path exceeds the Hubble distance. The numbers are roughly

$$\begin{aligned} T_{\rm dec} &\sim 3000 \text{ K} \sim 0.26 \text{ eV} \\ z_{\rm dec} &\sim 1100. \end{aligned}$$

The decoupling means that the recombination reaction can no more keep the ionisation fraction on the equilibrium track, but instead we are left with a residual ionisation of $x \sim 10^{-4}$.

A long time later (at $z \sim 10$) the first stars form, and their radiation *reionises* the gas that is left in interstellar space. The gas has now such a low density, however, that the universe remains transparent.

Exercise: Transparency of the universe. We say the universe is transparent when the photon mean free path λ_γ is larger than the Hubble length $l_H = H^{-1}$, and opaque when $\lambda_\gamma < l_H$. The photon mean free path is determined mainly by the scattering of photons by free electrons, so that $\lambda_\gamma = 1/(\sigma_T n_e)$, where $n_e = x n_e^*$ is the number density of free electrons, n_e^* is the total number density of electrons, and x is the ionisation fraction. The cross section for photon-electron scattering is independent of energy for $E_\gamma \ll m_e$ and is then called the Thomson cross section, $\sigma_T = \frac{8\pi}{3}(\alpha/m_e)^2$, where α is the fine-structure constant. In recombination x falls from 1 to 10^{-4}. Show that the universe is opaque before recombination and transparent after recombination. (Assume the recombination takes place between instantly at $z = 1300$. You can assume a matter-dominated universe—see below for parameter values.) The interstellar matter gets later reionised (to $x \sim 1$) by the

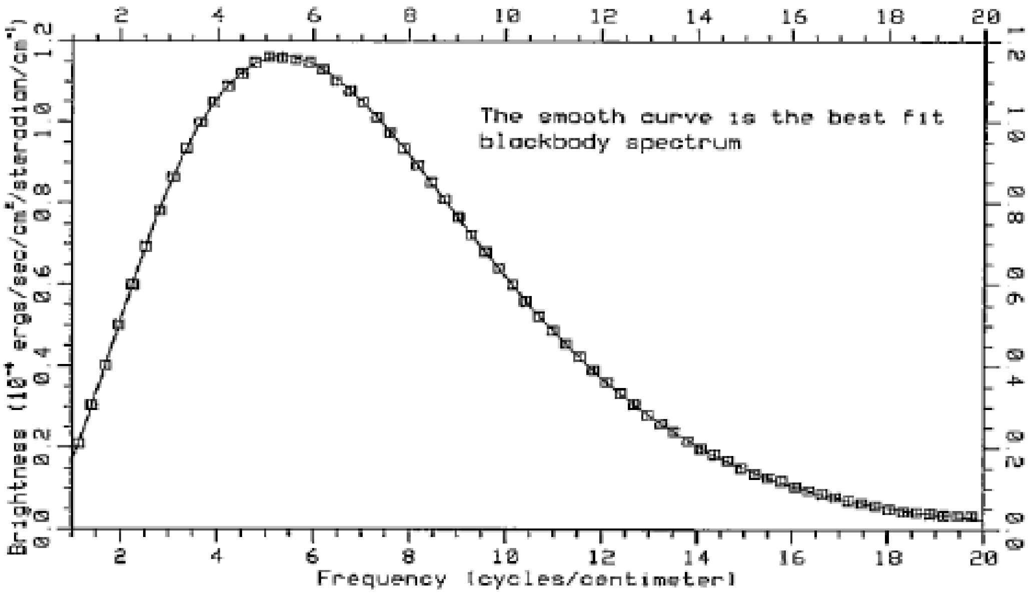

Figure 5: The CMB frequency spectrum as measured by the FIRAS instrument on the COBE satellite [3]. This first spectrum from FIRAS is based on just 9 minutes of measurements. The CMB temperature estimated from it was $T = 2.735 \pm 0.060$ K. The final result from FIRAS is $T = 2.725 \pm 0.002$ K (95% confidence) [4]. Using data from other experiments as well, the best current value is $T_0 = 2.72548 \pm 0.00057$ K (68% confidence) [5].

light from the first stars. What is the earliest redshift when this can happen without making the universe opaque again? (You can assume that most ($\sim$ all) matter has remained interstellar.) Calculate for $\Omega_{m0} = 1.0$ and $\Omega_{m0} = 0.3$ (note that Ω_m includes nonbaryonic matter). Use $\Omega_\Lambda = 0$, $h = 0.7$ and $\eta = 6 \times 10^{-10}$.

The photons in the cosmic background radiation have thus travelled almost without scattering through space all the way since we had $T = T_{\rm dec} \sim 1090\, T_0$.[7] When we look at this cosmic background radiation we thus see the universe (its faraway parts near our horizon) as it was at that early time. Because of the redshift, these photons which were then largely in the visible part of the spectrum, have now become microwave photons, so this radiation is now called the *cosmic microwave background* (CMB). It still maintains the thermal equilibrium distribution. This was confirmed to high accuracy by the FIRAS (Far InfraRed Absolute Spectrophotometer) instrument on the COBE (Cosmic Background Explorer) satellite in 1989. John Mather received the 2006 Physics Nobel Prize for this measurement of the CMB frequency (photon energy) spectrum, see figure 5.[8]

We shall now, for a while, stop the detailed discussion of recombination and photon decoupling. The universe is about 380 000 years old at decoupling. Next, gravitationally bound structures start to form as gravity attracts matter into overdense regions. Before photon decoupling the radiation pressure from photons pre-

[7]The probability for a photon to have one or more scatterings between decoupling and today is about 10%.

[8]He shared the prize with George Smoot, who got it for the discovery of the CMB anisotropy with the DMR instrument on the same satellite. We will discuss the CMB anisotropy in the second part of the course.

vented this. But before going to the physics of *structure formation*, we discuss some earlier events (big bang nucleosynthesis, dark matter decoupling and inflation) in more detail.

5.7 The Dark Ages

How would the universe after recombination appear to an observer with human eyes? At first one would see a uniform glow everywhere, since the wavelengths of many of the CMB photons are in the visible range, though the peak is in the infrared. (It would also feel rather hot, 3000 K). As time goes on, this glow gets dimmer and dimmer as the photons redshift towards the infrared, and after a few million years it gets completely dark, as photons even deep into the tail of the Planck distribution are redshifted into the infrared. There are no stars yet. This era is often called the *Dark Ages* of the universe. It lasts dozens of millions of years. While it lasts, it gets colder and colder. In the dark, however, masses are gathering together. And then, one by one, the first stars light up.

It seems that the starformation rate peaked between redshifts $z = 1$ and $z = 2$. Thus the universe at a few billion years was brighter than it is today, since the brightest stars are short-lived, and the galaxies were closer to each other back then.[9]

5.8 The radiation and neutrino backgrounds

While the starlight is more visible to us than the cosmic microwave background, it's average energy and photon number density in the universe is much less. Thus the photon density is essentially given by the CMB. The number density of CMB photons today ($T_0 = 2.725$ K) is

$$n_{\gamma 0} = \frac{2\zeta(3)}{\pi^2} T_0^3 = 410.5 \text{ photons/cm}^3 \tag{5.19}$$

and the energy density is

$$\rho_{\gamma 0} = \frac{\pi^2}{15} T_0^4 = 2.701 T_0 n_{\gamma 0} = 4.641 \times 10^{-31} \text{kg/m}^3 \,. \tag{5.20}$$

Since the critical density today is

$$\rho_{c0} = \frac{3H_0^2}{8\pi G} = h^2 \cdot 1.8788 \times 10^{-26} \text{kg/m}^3 \tag{5.21}$$

we get for the photon density parameter

$$\Omega_{\gamma 0} \equiv \frac{\rho_{\gamma 0}}{\rho_{c0}} = 2.47 \times 10^{-5} h^{-2} \,. \tag{5.22}$$

While relativistic, neutrinos contribute another radiation component

$$\rho_\nu = \frac{7N_\nu}{8} \frac{\pi^2}{15} T_\nu^4 \,. \tag{5.23}$$

[9] Though note that galaxies seen from far away are rather faint objects, difficult to see with the unaided eye. If you were suddenly transported to a random location in the present universe, you might not be able to see anything. To enjoy the spectacle, our hypothetical observer should be located within a forming galaxy, or have a good telescope.

After e^+e^- annihilation this gives

$$\rho_\nu = \frac{7N_\nu}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\rho_\gamma \ , \tag{5.24}$$

where N_ν is the number of neutrino species.

When the number of (light) neutrino species was not yet known from colliders, cosmology was used to constrain it. Big bang nucleosynthesis is sensitive to the expansion rate in the early universe, and that depends on the energy density. Observations require $2.7 < N_\nu < 3.1$ [6] (though this limit is somewhat dependent on the precise assumptions about the cosmological model). Actually any new particle species that would be relativistic around nucleosynthesis ($T \sim 50$ keV – 1 MeV) and would thus contribute to the expansion rate through its energy density, but which would not interact directly with nuclei and electrons, would have the same effect. The presence of such unknown particles at BBN is thus rather constrained.

If we take (5.24) to define N_ν, but then take into account the extra contribution to ρ_ν from energy leakage during e^+e^--annihilation (and some other small effects), we get (as a result of years of hard work by many theorists)

$$N_\nu = 3.046 \ . \tag{5.25}$$

(This does not mean that there are 3.046 neutrino species, but that the total energy density in neutrinos is 3.046 times as much as the energy density one neutrino species would contribute had it decoupled completely before e^+e^- annihilation.)

If neutrinos were still relativistic today, the neutrino density parameter would be

$$\Omega_{\nu 0} = \frac{7N_\nu}{22}\left(\frac{4}{11}\right)^{\frac{1}{3}}\Omega_{\gamma 0} = 1.71\times 10^{-5}h^{-2} \ , \tag{5.26}$$

so the total radiation density parameter would be

$$\Omega_{r0} = \Omega_{\gamma 0} + \Omega_{\nu 0} = 4.18\times 10^{-5}h^{-2} \sim 10^{-4} \ . \tag{5.27}$$

We thus confirm the claim in chapter 3 that the radiation component can be ignored in the Friedmann equation, except in the early universe. The combination $\Omega_i h^2$ is denoted by ω_i, so we have

$$\omega_\gamma = 2.47\times 10^{-5} \tag{5.28}$$

$$\omega_\nu = 1.71\times 10^{-5} \tag{5.29}$$

$$\omega_r = \omega_\gamma + \omega_\nu = 4.18\times 10^{-5} \, . \tag{5.30}$$

As noted earlier, neutrinos have masses in the meV to eV range. Thus neutrinos are nonrelativistic today and count as matter, not radiation, so the above result for the neutrino energy density does not apply. However, unless the neutrino masses are above 0.2 eV, they would still have been relativistic, and counted as radiation, at the time of recombination and matter-radiation equality. While the neutrinos are relativistic, we get neutrino energy density

$$\rho_\nu = \Omega_{\nu 0}\rho_{c0}a^{-4} \tag{5.31}$$

using $\Omega_{\nu 0}$ from (5.26), even though $\Omega_{\nu 0}$ does not give the present density of neutrinos.

Today, even though the photon and neutrino backgrounds do not dominate the energy density of the universe any more, they do dominate the entropy density.

Exercise: Matter–radiation equality. The present density of matter is $\rho_{m0} = \Omega_{m0}\rho_c$ and the present density of radiation is $\rho_{r0} = \rho_{\gamma 0} + \rho_{\nu 0}$ (we assume neutrinos are massless). What was the age of the universe $t_{\rm eq}$ when $\rho_m = \rho_r$? (Note that in these early times—but not today—you can ignore the curvature and vacuum terms in the Friedmann equation.) Give numerical value (in years) for the cases Ω_{m0} = 0.1, 0.3, and 1.0, and H_0 = 70 km/s/Mpc. What was the temperature at that time, $T_{\rm eq}$?

References

[1] M. D'Onofrio and K. Rummukainen, arXiv:1508.07161 [hep-ph].

[2] K.A. Olive et al. (Particle Data Group), Chin. Phys. C, 38 (2014) 090001, http://pdg.lbl.gov/2014/listings/contents_listings.html

[3] J.C. Mather et al., Astrophys. J. Lett. **354** (1990) 37.

[4] J.C. Mather et al., Astrophys. J. **512** (1999) 511.

[5] D.J. Fixsen, Astrophys. J. **707** (2009) 916.

[6] G. Rossi, C. Yeche, N. Palanque-Delabrouille and J. Lesgourgues, arXiv:1412.6763 [astro-ph.CO].