9 Linear perturbation theory

9.1 Structure formation

Up to now we have discussed the universe in terms of the homogeneous and isotropic FRW model. We have, however, used the notion of temperature, which involves fluctuations, so inhomogeneities have already implicitly been present. We now take the next step by explicitly considering small perturbations around the homogeneous and isotropic model (which we refer to as the "unperturbed" or "background" universe). In cosmology, perturbation theory has wide applicability. Very non-linear structures such as planets, stars and galaxies have evolved from small initial perturbations under the influence of gravity, and often the distribution of non-linear objects can be treated in terms of linear theory, even though their internal composition cannot. The growth of non-linear structures is called *structure formation*. The discussion of perturbations can be divided into two parts.

1) The generation of the primordial perturbations, "the seeds of structure". This is the more speculative part of structure formation theory. We don't know beyond reasonable doubt how the primordial perturbations were generated, but we have an excellent candidate scenario, *inflation*, the predictions of which have agreed very well with observations, and which are being tested more thoroughly. According to the inflationary scenario, all structure originates from *quantum fluctuations* in the early universe.

2) The growth of the small perturbations into the present observable structure of the universe. This part is less speculative, since we have a well established theory of gravity, general relativity. However, there is uncertainty in this part too, since we do not know the precise nature of the dominant components to the energy density of the universe, the *dark matter* and the possible *dark energy.* The gravitational growth depends on the equations of state and the streaming lengths (particle mean free path between interactions) of these density components. Besides gravity, the growth is affected by pressure (due to non-gravitational interactions).

We will first discuss the formalism of cosmological perturbation theory. We will apply it to the generation and early evolution of structures, then to the evolution of the perturbations in the various later eras in the history of the universe. We will then discuss the cosmic microwave background using perturbation theory. We will not discuss the formation of galaxies or other non-linear structures except in very general terms, as we only follow perturbations up to the time when they enter the non-linear regime.

We will work with *first order perturbation theory* (also called *linear perturbation theory*). This means that all quantities are written as a sum of the background value, corresponding to the homogeneous and isotropic model, and a perturbation, which is the deviation from the background value, and we neglect all terms that are higher than first order in the perturbations. For example, for the energy density we have

$$\rho(t, \boldsymbol{x}) = \bar{\rho}(t) + \delta\rho(t, \boldsymbol{x}) \ ,$$

where $\boldsymbol{x}$ are the comoving spatial coordinates. As we drop all terms that contain a product of two or more perturbations, the remaining equations contain only terms which are either *zeroth order*, i.e. contain only background quantities, or *first order*, i.e. contain exactly one power of the perturbed quantities. If we understand the zeroth order parts as the average, then the average of the perturbations vanishes. By averaging the inhomogeneous equations we thus get back the equations of the homogeneous and isotropic universe. Subtracting these from our equations we arrive at the *perturbation equations* where every term is first order in the perturbation quantities, i.e. the equations are linear[1].

9.2 The perturbed metric

Let us first discuss perturbations of the metric. We leave the rigorous development of cosmological perturbation theory to a more advanced course, and just summarise some basic concepts and results. (The interested reader may consult [1, 2] for details.) We have the line element

$$\begin{aligned} \mathrm{d}s^2 &= g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\ &= (\bar{g}_{\alpha\beta} + \delta g_{\alpha\beta})\mathrm{d}x^\alpha \mathrm{d}x^\beta \ , \end{aligned} \tag{9.1}$$

where $\bar{g}_{\alpha\beta}$ is the background metric,

$$\bar{g}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i \mathrm{d}x^j \ , \tag{9.2}$$

and $\delta g_{\alpha\beta}$ is a perturbation, which we take to be small. In this course, we only consider spatially flat backgrounds, as spatial curvature would introduce technical complications and inflation is expected to make the spatial curvature tiny. The question what is a small perturbation is not entirely straightforward. For example, we might naively demand $|\delta g_{\alpha\beta}| \ll |\bar{g}_{\alpha\beta}|$. But (leaving aside that some of the components $\bar{g}_{\alpha\beta}$ are zero) this kind of a statement is coordinate-dependent. We can make a coordinate transformation that will make a large change to the metric, while keeping the physics exactly the same. An example would be a large Lorentz boost. This shows another problem, namely that perturbations in the metric do not necessarily correspond to changes in the physical state of the system.

In general, if perturbations in all physical quantities are small, it should be possible to choose a coordinate system where the metric perturbations are small (compared to unity). Note that the reverse is not true: from the fact that the metric perturbations are small one cannot conclude that perturbations in all physical quantities are small. For example, the gravitational field in the solar system is quite small, and the solar system can be represented as a linear perturbation around Minkowski space. However, the energy density in the solar system changes by a factor 10^{20} when going from Earth to interplanetary space.

From now on, we assume that we have chosen an appropriate coordinate system such that the metric perturbations are small, so we can neglect all terms which are second order or higher in the metric perturbations. In the linear approximation, the metric perturbations do not influence the evolution of the background on which they live.

The metric perturbations inherit geometric structure from the background. Just like in classical electrodynamics we can decompose a general tensor into irreducible

[1] This way of decoupling the background and the perturbations does not work straightforwardly beyond first order perturbation theory. We will be content with linear theory.

representations of the Lorentz group, we can decompose the metric perturbations into irreducible parts with regard to the symmetries of the background, namely translation and rotation in the spatial dimensions. In less technical language, the perturbations can be split up into things which have either zero, one or two spatial indices, and which we can treat like scalars, vectors and tensor living on a Euclidean space. The most general linear perturbation around the FRW metric (9.2), decomposed into its irreducible parts, reads

$$\begin{aligned} \mathrm{d}s^2 &= g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\ &= \bar{g}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + \delta g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\ &= -(1+2\Phi)\mathrm{d}t^2 + 2a(t)(B_{,i} - S_i)\mathrm{d}x^i \mathrm{d}t \\ &\quad + a(t)^2 \left[(1-2\Psi)\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}\right] \mathrm{d}x^i \mathrm{d}x^j \ , \end{aligned} \tag{9.3}$$

where Φ, Ψ, B and E are *scalars*, S_i, F_i are *vectors* and h_{ij} is a *tensor*, and a comma stands for derivative with respect to x^i i.e. $f_{,i} \equiv \partial f/\partial x^i$. The vector perturbations are transverse, $\delta^{ij} S_{i,j} = \delta^{ij} F_{i,j} = 0$, and the tensor perturbation is transverse and traceless, $\delta^{ij} h_{ij} = 0, h_{ij,j} = 0$. Physically, tensors correspond to gravity waves, vectors describe rotation and scalars are directly related to the density perturbation, as we will see.

Since we drop all non-linear terms, the scalar, vector and tensor perturbations evolve independently. The vector perturbations decay with the expansion, so they are expected to be negligible in the linear regime, and we put them to zero, $S_i = F_i = 0$. There can be significant tensor perturbations in the universe, and they may be observable in the cosmic microwave background anisotropy. The amplitude of the perturbations depends on the details of inflation. No tensor perturbations have been detected thus far[2].

For the metric perturbation, we have 10 functions $\delta g_{\alpha\beta}(t, \boldsymbol{x})$. So there would appear to be ten degrees of freedom. However, four of them are not physical degrees of freedom, they just correspond to the freedom of choosing the four coordinates. So there are 6 physical degrees of freedom. There are thus different coordinate systems (also called different *gauges*) which describe the same physics. The choice of coordinates is called a choice of gauge[3]. It can be shown that we can choose $E = B = 0$, and that doing so fixes the coordinate system completely. This choice is known as the *longitudinal gauge* and also as *the conformal Newtonian gauge.* We are then left with the metric

$$\begin{aligned} \mathrm{d}s^2 &= g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\ &= \bar{g}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + \delta g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\ &= -(1+2\Phi)\mathrm{d}t^2 + a(t)^2 \left[(1-2\Psi)\delta_{ij} + h_{ij}\right] \mathrm{d}x^i \mathrm{d}x^j \ , \end{aligned} \tag{9.4}$$

so we have two scalar degrees of freedom and one transverse traceless symmetric tensor, which has two independent degrees of freedom. The metric perturbations

[2] There was a claim of detection by the BICEP2 team in 2014, but the claimed signal turned out to be due to foreground contamination by Galactic dust.

[3] More precisely, perturbation theory is formulated in terms of a mapping from the real inhomogeneous and anisotropic spacetime to a background spacetime, and it is the choice of map which is called a "gauge choice". However, the choice of coordinates and choice of mapping are often conflated in cosmological parlance. More simply, change of gauge is a change of coordinates, except that it only affects the perturbations, the background is kept fixed. We will not get into such details.

$\Phi(t, \boldsymbol{x})$ and $\Psi(t, \boldsymbol{x})$ are called the *Bardeen potentials*[4]. The function Φ is also called the Newtonian potential, since in the Newtonian limit, it becomes equal to the Newtonian potential perturbation, and Ψ is called the Newtonian curvature perturbation, because it determines the curvature of the 3-dimensional $t = \text{const.}$ subspaces, which are flat in the unperturbed universe.

The evolution of the metric perturbations is determined by the Einstein equation, which couples the metric to the matter content as described by the energy-momentum tensor.

9.3 The perturbed equations of motion

The Einstein equation is

$$G_{\alpha\beta} = 8\pi G_{\mathrm{N}} T_{\alpha\beta} \ , \tag{9.5}$$

where $G_{\alpha\beta}$ is a tensor which is built from the metric and its first and second derivatives, and the energy-momentum tensor $T_{\alpha\beta}$ describes the properties of matter. In chapter 3 we noted that for an ideal fluid the energy-momentum tensor has the following form

$$T_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + p g_{a\beta} \ , \tag{9.6}$$

where ρ is the energy density and p is the pressure measured by an observer moving with four-velocity u^α. In the FRW case, the energy-momentum tensor necessarily has this form for all forms of matter due to the symmetry of the spacetime. In the perturbed case, the energy-momentum tensor can also have contributions from *energy flux* and *anisotropic stress* in addition to then energy density and pressure. We will not discuss imperfect fluids here.

As with the metric, we split the contributions to the energy-momentum tensor into background plus perturbations,

$$\begin{aligned} \rho(t, \boldsymbol{x}) &= \bar{\rho}(t) + \delta\rho(t, \boldsymbol{x}) && (9.7) \\ p(t, \boldsymbol{x}) &= \bar{p}(t) + \delta p(t, \boldsymbol{x}) && (9.8) \\ u^\alpha(t, \boldsymbol{x}) &= \delta^{\alpha 0} + \delta u^\alpha(t, \boldsymbol{x}) \ , && (9.9) \end{aligned}$$

and we throw out all terms which have two or more powers of the perturbations, whether of the metric or the matter variables. The four-velocity is normalised as $g_{\alpha\beta} u^\alpha u^\beta = -1$, from which it follows that $\delta u^0 = -\Phi$ in linear theory.

Equating the Einstein tensor corresponding to the metric (9.4) to the energy-momentum tensor (9.6) (times $8\pi G_{\mathrm{N}}$) in the linear approximation, we get the familiar equations for the background:

$$\begin{aligned} 3H^2 &= 8\pi G_{\mathrm{N}} \bar{\rho} && (9.10) \\ 3(\dot{H} + H^2) &= -4\pi G_{\mathrm{N}} (\bar{\rho} + 3\bar{p}) \ , && (9.11) \end{aligned}$$

[4]Warning: Sign conventions for Φ and Ψ differ, and the definitions of Ψ and Φ are also sometimes switched with each other.

where we have used the relation $\ddot{a}/a = \dot{H} + H^2$. For the perturbations, we get

$$
\begin{aligned}
4\pi G_{\rm N}\delta\rho &= \frac{1}{a^2}\nabla^2\Psi - 3H(\dot{\Psi} + H\Phi) && (9.12)\\
4\pi G_{\rm N}(\bar{\rho}+\bar{p})\delta u_i &= -(\dot{\Psi} + H\Phi)_{,i} && (9.13)\\
4\pi G_{\rm N}\delta p\delta_{ij} &= \left[(2\dot{H} + 3H^2)\Phi + H\dot{\Phi} + \ddot{\Psi} + 3H\dot{\Psi} + \frac{1}{2}\frac{1}{a^2}\nabla^2 D\right]\delta_{ij} \\
&\quad -\frac{1}{2}\frac{1}{a^2}D_{,ij} && (9.14)\\
0 &= \ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij}\ , && (9.15)
\end{aligned}
$$

where $\nabla^2 \equiv \delta^{ij}\partial_i\partial_j$ and $D \equiv \Phi - \Psi$. These are the central equations for discussing the evolution of perturbations. In this course, we cannot properly derive them from the general Einstein equation, we have to take them as given.

From the non-diagonal components of (9.14) we get that $D_{,ij} = 0$ for all $i \neq j$. The general solution of this equation is $D = A(t,x) + B(t,y) + C(t,z)$. In cosmology there are no preferred coordinate axes, so the only physically relevant solution is $D = D(t)$. However, this corresponds to changing the time coordinate, so we can set $D(t) = 0$ without loss of generality. We therefore have $\Phi = \Psi$.[5] To see what the single remaining scalar metric degree of freedom corresponds to, we can manipulate the remaining perturbations equations (9.12)–(9.14). Let us introduce some notation: the *density contrast* is defined as

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}}\ . \tag{9.16}$$

We also define the background equation of state as $w \equiv \bar{p}/\bar{\rho}$, and introduce the variable $v^2 \equiv \delta p/\delta\rho$. We will later see that if $v^2 > 0$, then v corresponds (for certain types of perturbation called *adiabatic*) to the sound speed of the cosmic fluid; if $v^2 < 0$, it instead describes instability timescale of the fluid. We can now express the pressure perturbation in terms of v^2 and δ, and write (9.12)–(9.15) as

$$
\begin{aligned}
0 &= \ddot{\Phi} + H(4+3v^2)\dot{\Phi} - v^2\frac{1}{a^2}\nabla^2\Phi + [2\dot{H} + (3+3v^2)H^2]\Phi && (9.17)\\
\delta &= \frac{2}{3}\frac{1}{(aH)^2}\nabla^2\Phi - 2\frac{1}{H}\dot{\Phi} - 2\Phi && (9.18)\\
\delta u^i &= \frac{1}{a^2\dot{H}}\partial_i(\dot{\Phi} + H\Phi) && (9.19)\\
0 &= \ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij}\ . && (9.20)
\end{aligned}
$$

From the set of equations (9.17)–(9.19) it follows that the metric perturbation Φ is non-zero only if there is matter. So ϕ is generated directly by matter sources,

[5] In fact, neutrinos develop anisotropic stress after neutrino decoupling, they do not behave like an ideal fluid. Therefore the two Bardeen potentials actually differ from each other by about 10% in the time between neutrino decoupling and matter-radiation equality. After the universe becomes matter-dominated, the neutrinos become unimportant, and Ψ and Φ rapidly approach each other. The same thing happens to photons after photon decoupling, but the universe is then already matter-dominated, so the photons do not cause a significant difference between Ψ and Φ.

in particular by the density perturbations. In contrast, the tensor perturbation h_{ij} can be non-zero even if the space is empty: they correspond to *gravity waves.*

The procedure for solving the perturbed equations is the following.

1) Give the matter model, i.e. give w and v^2.

2) Solve for the evolution of the background and obtain $a(t)$.

3) Solve the perturbation equations.

The order of solving the perturbation equations is that (9.17) gives the evolution of Φ, and we then find the corresponding density contrast from (9.18) and the velocity perturbation from (9.19). (We will not be much concerned about the velocity perturbation.) Note an important difference in (9.18) from the classical Poisson equation: there are terms of the metric perturbation without any gradients on the right-hand side. This is a purely general relativistic feature which has very important consequences, as we will see.

9.4 Fourier transformation

Since the equations are linear, they are easily solved in terms of a Fourier transformation. We define

$$\Phi(t, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k \Phi_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ , \tag{9.21}$$

and $\delta_{\boldsymbol{k}}$, $u^i_{\boldsymbol{k}}$ and $h_{\boldsymbol{k}ij}$ are defined in the same way. Because the universe is expanding, the variable $\boldsymbol{k}$, called the *comoving momentum* or *comoving wavenumber*, is not the physical momentum, which is instead given by $\boldsymbol{k}/a$. With the scale factor normalised to unity today, the comoving momentum of a Fourier mode is the physical momentum it has today.

The flatness of the spatial sections is crucial here. If the spatial sections were curved, plane waves would not form a complete set of basis functions, and we would instead have to use more complicated functions. (There would also be an additional scale present, given by the spatial curvature term K/a^2.)

Different Fourier modes decouple, and the equations for the metric perturbations reduce to ordinary second order differential equations for each mode. Inserting (9.21) into (9.17)–(9.20) we get (we drop the velocity equation)

$$0 = \ddot{\Phi}_{\boldsymbol{k}} + H(4+3v^2)\dot{\Phi}_{\boldsymbol{k}} + v^2 \frac{k^2}{a^2}\Phi_{\boldsymbol{k}} + [2\dot{H} + (3+3v^2)H^2]\Phi_{\boldsymbol{k}} \tag{9.22}$$

$$\delta_{\boldsymbol{k}} = -\frac{2}{3}\frac{k^2}{(aH)^2}\Phi_{\boldsymbol{k}} - 2\frac{1}{H}\dot{\Phi}_{\boldsymbol{k}} - 2\Phi_{\boldsymbol{k}} \tag{9.23}$$

$$0 = \ddot{h}_{\boldsymbol{k}ij} + 3H\dot{h}_{\boldsymbol{k}ij} + \frac{k^2}{a^2} h_{\boldsymbol{k}ij} \ , \tag{9.24}$$

where we have denoted $k \equiv |\boldsymbol{k}|$.

The above equations have an interesting property. For a fluid for which $v^2 = w$, the last term in (9.22) vanishes due to (9.10) and (9.11). Thus, for long wavelength perturbations, $k \ll aH$, we find that $\Phi_{\boldsymbol{k}}$ =constant is a solution of the equations, and (9.23) shows that the density contrast $\delta_{\boldsymbol{k}}$ is then also constant in time and equal to $-2\Phi_{\boldsymbol{k}}$. The gravity waves also have a constant solution, regardless of v^2 or the equation of state, as long as $k \ll aH$. So the relativistic equations allow for the possibility that perturbations with wavelengths much larger than the Hubble scale are 'frozen in' and remain unaffected by cosmological evolution. Such a feature is not present in Newtonian gravity.

In the first part of the course we saw that the early universe is radiation-dominated until $t = t_{\text{eq}} \approx 50\,000$ years, after which the universe is matter-dominated until it becomes (in the ΛCDM model) dominated by the vacuum energy at around 8 billion years. In order to know the evolution of the perturbations, all we need to do is to plug the background evolution we have already calculated into the above equations and solve, keeping in mind that we have to track at least four different components (photons, neutrinos, baryons and dark matter) with different behaviour (i.e. different w and v^2).

The equations (9.22) and (9.23) give the time evolution of the Fourier components, but the spatial dependence (i.e. dependence on $\boldsymbol{k}$) is left unconstrained, and since the equations are linear, all linear combinations of solutions are also solutions. The spatial dependence is fixed by the initial conditions at early times. Until the 1980s, initial conditions were based on assumptions of simplicity, but today we have a scenario called *inflation* in which it is possible to actually calculate how perturbations are generated from quantum fluctuations. We will discuss this in the next chapter, but let us first consider some statistical properties of fluctuations.

9.5 Gaussian perturbations

Simplest models of inflation predict, and observations show, that cosmological perturbations are (in the linear regime) close to *Gaussian*. Possible deviations from Gaussianity are a topical subject in cosmology at the moment. No deviations in the primordial perturbations have been found, and the non-Gaussian contribution relative to the Gaussian contribution has to be less than 10^{-4}, according to observations by the Planck satellite. (Non-linear structure formation does destroy the Gaussianity of the initial perturbations on small scales.) Let us discuss a generic Gaussian perturbation $g(\boldsymbol{x})$, where g could be Φ, δ or some other linear theory quantity (we suppress the time dependence here):

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{k}} g_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ , \tag{9.25}$$

where the set of Fourier coefficients $\{g_{\boldsymbol{k}}\}$ is the result of a Gaussian random process. We have here used a Fourier series instead of the integral Fourier transformation. Formally, this corresponds to considering some cubic region ("box") of the universe, in the comoving coordinates, with some comoving volume L^3 and assuming periodic boundary conditions. The box is just a physically irrelevant convenient mathematical device. In the end we can take the limit $L^3 \to \infty$ and replace the Fourier series with a Fourier integral. (See section 9.6 for the correspondence.) In cosmology, we can only predict the probability distribution from which the perturbations are drawn (since they originate in a quantum process), not the particular realisation that corresponds to our universe. This causes some limitations on the comparison between theory and observation, as will see when we discuss the cosmic microwave background.

Cosmological perturbations are real, so we have $g_{-\boldsymbol{k}} = g^*_{\boldsymbol{k}}$. We can write $g_{\boldsymbol{k}}$ in terms of its real and imaginary part,

$$g_{\boldsymbol{k}} = \alpha_{\boldsymbol{k}} + i\beta_{\boldsymbol{k}} \,. \tag{9.26}$$

To know a random process means to know the *probability distribution* $\text{Prob}(g_{\boldsymbol{k}})$. The *expectation value* of a quantity which depends on $g_{\boldsymbol{k}}$ as $f(g_{\boldsymbol{k}})$ is given by

$$\langle f(g_{\boldsymbol{k}})\rangle \equiv \int f(g_{\boldsymbol{k}})\text{Prob}(g_{\boldsymbol{k}})d\alpha_{\boldsymbol{k}} d\beta_{\boldsymbol{k}} \,, \tag{9.27}$$

where the integral is over the complex plane, i.e.

$$\int_{-\infty}^{\infty} d\alpha_{\boldsymbol{k}} \int_{-\infty}^{\infty} d\beta_{\boldsymbol{k}} \,.$$

We now define what we mean by *Gaussian perturbations* (or by a *Gaussian random process*, a process that produces such perturbations). We restrict to perturbations with zero mean, which is the relevant situation in cosmology. Such perturbations $g(\boldsymbol{x})$ satisfy two properties:

1. The probability distribution of an individual Fourier component is Gaussian[6]:

$$\begin{aligned} \text{Prob}(g_{\boldsymbol{k}}) &= \frac{1}{2\pi s_{\boldsymbol{k}}^2} \exp\left(-\frac{1}{2}\frac{|g_{\boldsymbol{k}}|^2}{s_{\boldsymbol{k}}^2}\right) \\ &= \frac{1}{\sqrt{2\pi} s_{\boldsymbol{k}}} \exp\left(-\frac{1}{2}\frac{\alpha_{\boldsymbol{k}}^2}{s_{\boldsymbol{k}}^2}\right) \times \frac{1}{\sqrt{2\pi} s_{\boldsymbol{k}}} \exp\left(-\frac{1}{2}\frac{\beta_{\boldsymbol{k}}^2}{s_{\boldsymbol{k}}^2}\right) \,. \end{aligned} \tag{9.28}$$

   From this distribution we immediately get (**exercise**) its *mean*

$$\langle g_{\boldsymbol{k}}\rangle = 0 \tag{9.29}$$

   and *variance*

$$\langle |g_{\boldsymbol{k}}|^2\rangle = 2s_{\boldsymbol{k}}^2 \,. \tag{9.30}$$

   The distribution has one free parameter for each value of $\boldsymbol{k}$, the real positive number $s_{\boldsymbol{k}}$ that gives the width (determines the variance) of the distribution.

2. The probabilities of different Fourier modes are independent (i.e., they are not correlated),

$$\langle g_{\boldsymbol{k}} g^*_{\boldsymbol{k}'}\rangle = 0 \qquad \text{for} \quad \boldsymbol{k} \neq \boldsymbol{k}' \,. \tag{9.31}$$

   Because of the *, this holds also when $\boldsymbol{k}' = -\boldsymbol{k}$ (exercise).

In addition, the distribution is assumed to be *statistically homogeneous and isotropic* in space. This means that the probability distribution is independent of the direction of the Fourier mode $\boldsymbol{k}$:

$$s_{\boldsymbol{k}} = s(k) \,. \tag{9.32}$$

Like Gaussianity, this is a prediction of typical models of inflation, and seems to be agreement with the data. (There appear to be some anomalies in the CMB which may point to a small violation of this symmetry, but the issue remains unsettled.)

[6] We take the definition of Gaussianity to include zero mean.

We can combine (9.30) and (9.31) into a single equation,

$$\langle g_{\boldsymbol{k}} g^*_{\boldsymbol{k}'} \rangle = 2\delta_{\boldsymbol{k}\boldsymbol{k}'} s^2_{\boldsymbol{k}} = \delta_{\boldsymbol{k}\boldsymbol{k}'} \langle |g_{\boldsymbol{k}}|^2 \rangle \tag{9.33}$$

Going from Fourier space back to coordinate space, we find

$$\langle g(\boldsymbol{x}) \rangle = \left\langle \sum_{\boldsymbol{k}} g_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \right\rangle = \sum_{\boldsymbol{k}} \langle g_{\boldsymbol{k}} \rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0 \tag{9.34}$$

The expectation value of the perturbation is zero, since it represents a deviation from the background value. The square of the perturbation can be written as

$$g(\boldsymbol{x})^2 = \sum_{\boldsymbol{k}} g^*_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \sum_{\boldsymbol{k}'} g_{\boldsymbol{k}'} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} \tag{9.35}$$

since $g(\boldsymbol{x})$ is real. The typical amplitude of the perturbation is described by the variance, the expectation value of this square,

$$\langle g(\boldsymbol{x})^2 \rangle = \sum_{\boldsymbol{k}\boldsymbol{k}'} \langle g^*_{\boldsymbol{k}} g_{\boldsymbol{k}'} \rangle e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{x}} = \sum_{\boldsymbol{k}} \langle |g_{\boldsymbol{k}}|^2 \rangle = 2 \sum_{\boldsymbol{k}} s^2_{\boldsymbol{k}} \,. \tag{9.36}$$

9.6 The power spectrum

As noted above, all statistical information about a Gaussian perturbation is encoded in a single function of one variable. In cosmology, this function gives the spatial dependence of the initial conditions for the perturbations, and it is usually discussed in terms of the *power spectrum*, which is defined as

$$\mathcal{P}_g(k) \equiv \left(\frac{L}{2\pi}\right)^3 4\pi k^3 \langle |g_{\boldsymbol{k}}|^2 \rangle = \frac{L^3}{2\pi^2} k^3 \langle |g_{\boldsymbol{k}}|^2 \rangle \,. \tag{9.37}$$

We will want to convert the Fourier series back into a Fourier integral. The correspondence between the two is (following Liddle & Lyth [3])

$$\begin{aligned} g(\boldsymbol{x}) &= \frac{1}{(2\pi)^{3/2}} \int g(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} d^3k \\ g(\boldsymbol{k}) &= \frac{1}{(2\pi)^{3/2}} \int g(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} d^3x \,. \end{aligned} \tag{9.38}$$

To take the limit of infinite box size, $L^3 \to \infty$, we replace

$$\begin{aligned} \left(\frac{2\pi}{L}\right)^3 \sum_{\boldsymbol{k}} &\quad\to\quad \int d^3k \\ \left(\frac{L}{2\pi}\right)^3 g_{\boldsymbol{k}} &\quad\to\quad \frac{1}{(2\pi)^{3/2}} g(\boldsymbol{k}) \\ \left(\frac{L}{2\pi}\right)^3 \delta_{\boldsymbol{k}\boldsymbol{k}'} &\quad\to\quad \delta^3(\boldsymbol{k}-\boldsymbol{k}') \end{aligned} \tag{9.39}$$

It is usually easiest to work with the series, and convert to the integral near the end (to avoid dealing with products of delta functions).

We find for the variance of $g(\boldsymbol{x})$,[7]

$$\begin{aligned}\langle g(\boldsymbol{x})^2\rangle &= \sum_{\boldsymbol{k}}\langle|g_{\boldsymbol{k}}|^2\rangle = \left(\frac{2\pi}{L}\right)^3\sum_{\boldsymbol{k}}\frac{1}{4\pi k^3}\mathcal{P}_g(k) \\ &\rightarrow \quad \frac{1}{4\pi}\int\frac{d^3k}{k^3}\mathcal{P}_g(k) = \int_0^\infty\frac{dk}{k}\mathcal{P}_g(k) = \int_{-\infty}^{\infty}\mathcal{P}_g(k)d\ln k\,.\end{aligned} \tag{9.40}$$

Thus the power spectrum of g gives the contribution of a logarithmic scale interval to the variance of $g(\boldsymbol{x})$. For Gaussian perturbations, the power spectrum gives a complete statistical description, and all statistical quantities can be calculated from it.

In practice the integration is not extended all the way from $k = 0$ to $k = \infty$. Rather, there is usually some largest and smallest relevant scale, and they introduce natural a cutoff at both ends of the integral. The largest relevant scale could be the size of the observable universe: The perturbation $g(\boldsymbol{x})$ represents a deviation from the background quantity, but the best estimate we have for the background may be the average taken over the observable universe. Then perturbations at larger scales contribute to our estimate of the background value instead of contributing to the perturbation away from it. The appropriate cutoff scale depends on the issue under study, and it is often essential to discuss also perturbations on scales larger than the Hubble scale. The smallest relevant scale in the present context is the end of the linear regime. By including non-linear corrections, it is possible to consider the power spectrum also in the non-linear regime, though on very small scales the original information has now been erased by non-linear processes. From a fundamental point of view, there is expected to be no primordial information left on very small scales anyway, because of the process of *free-streaming*, which we will discuss later. From a practical point of view, the relevant scale for comparing to observations it limited by the resolution of the observational survey considered. For example, if we consider density perturbations in terms of perturbations in the number density of galaxies, then this is only meaningfully defined on scales larger than the galaxy scales.

An alternative definition for the power spectrum is

$$P_g(k) \equiv L^3\langle|g_{\boldsymbol{k}}|^2\rangle\ . \tag{9.41}$$

Both this and the previous definition are used; in these notes we distinguish them by the different typeface. They are related by

$$P_g(k) = \frac{2\pi^2}{k^3}\mathcal{P}_g(k)\ . \tag{9.42}$$

Given the matter content and the initial condition in terms of the power spectrum (both for the scalar and tensor perturbations), the solution in the linear regime is completely determined by (9.20), (9.22) and (9.23). In the next chapter, we discuss how the initial field of Gaussian perturbations is generated by inflation and what are the expected power spectra for scalar and tensor perturbations.

[7] Note that the result has no $\boldsymbol{x}$-dependence. Even though the function $g(\boldsymbol{x})^2$ varies from place to place, its expectation value is the same everywhere.

References

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