10 Inflation: perturbations

10.1 The evolution of perturbations

10.1.1 The equations of motion

We now want to find out how perturbations are generated during inflation and how they evolve. In chapter 9 we gave the equations of motion for the metric perturbations, and noted that in order to solve them we need to give the background equation of state and $v^2 = \delta p/\delta\rho$. We have discussed the background evolution during inflation in chapter 8. However, rather than dealing with the perturbation equations in terms of the energy density and pressure, in the inflationary case it is more convenient to discuss perturbations in the inflaton field. As with the other quantities, we split the field into the background and the perturbation,

$$\varphi(t, \boldsymbol{x}) = \bar{\varphi}(t) + \delta\varphi(t, \boldsymbol{x}) \ . \tag{10.1}$$

In chapter 8, we derived the equation of motion for the scalar field,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) - V'(\varphi) = 0 \ . \tag{10.2}$$

In the spatially flat Friedmann-Robertson-Walker universe, we have

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + V'(\varphi) = 0 \ , \tag{10.3}$$

which in Minkowski space reduces to

$$\ddot{\varphi} - \nabla^2\varphi + V'(\varphi) = 0 \ . \tag{10.4}$$

We now input, instead of the FRW metric, the perturbed metric in the longitudinal gauge from chapter 9 into (10.2). We then get (recall that $g^{\mu\nu}$ is the inverse of the metric tensor)

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + \left(-\frac{1}{a^2}\nabla^2 + V''(\bar{\varphi})\right)\delta\varphi = -2\Phi V'(\bar{\varphi}) + \left(\dot{\Phi} + 3\dot{\Psi}\right)\dot{\bar{\varphi}} \ . \tag{10.5}$$

Making a Fourier transformation, we obtain

$$\delta\ddot{\varphi}_{\boldsymbol{k}} + 3H\delta\dot{\varphi}_{\boldsymbol{k}} + \left[\left(\frac{k}{a}\right)^2 + m^2(\bar{\varphi})\right]\delta\varphi_{\boldsymbol{k}} = -2\Phi_{\boldsymbol{k}} V'(\bar{\varphi}) + \left(\dot{\Phi}_{\boldsymbol{k}} + 3\dot{\Psi}_{\boldsymbol{k}}\right)\dot{\bar{\varphi}} \ . \tag{10.6}$$

where we have used $m^2(\bar{\varphi}) \equiv V''(\bar{\varphi})$.

We could now write the perturbed energy density and pressure of the scalar field, plug them into the perturbation equations given in chapter 9, and solve them in connection with (10.6). However, there is an easier way. We mentioned in chapter 9 that not all metric perturbations correspond to changes in physics. This is not just a nuisance, it may also be used for benefit. By making coordinate transformations (or more precisely gauge transformations!), we can change the form of our equations of motion to be more easily solved. Dealing with the details of the gauge transformations is beyond the scope of this course, so we just note that it is possible to choose the coordinate system so that metric perturbations make a negligible contribution to the equation of motion of the inflaton perturbations *during slow-roll inflation*, to first order in the slow-roll parameters[1]. The equation (10.6) then reduces to

$$\delta\ddot{\varphi}_{\boldsymbol{k}} + 3H\delta\dot{\varphi}_{\boldsymbol{k}} + \left[\left(\frac{k}{a}\right)^2 + m^2(\bar{\varphi})\right]\delta\varphi_{\boldsymbol{k}} = 0\ . \tag{10.7}$$

This is precisely what we would get if we just inserted (10.1) into the background equation of motion for the inflaton field and subtracted the background (i.e. ignored perturbations in the metric).

10.1.2 Solutions

During inflation, H and m^2 change slowly. Thus, we now make an approximation where we treat them as constants. The general solution of (10.7) is then

$$\delta\varphi_{\boldsymbol{k}}(t) = a^{-3/2}\left[A_{\boldsymbol{k}}J_{-\nu}\left(\frac{k}{aH}\right) + B_{\boldsymbol{k}}J_{\nu}\left(\frac{k}{aH}\right)\right]\ , \tag{10.8}$$

where J_ν is the Bessel function of order ν, with

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}\ . \tag{10.9}$$

The time dependence of the scale factor for constant H is

$$a(t) \propto e^{Ht}\ . \tag{10.10}$$

If the slow-roll approximation is valid, the inflaton has negligible mass, $m^2 \ll H^2$, since

$$\frac{m^2}{H^2} = 3M_{\text{Pl}}^2\frac{V''}{V} = 3\eta \ll 1\,. \tag{10.11}$$

Thus we can drop m^2/H^2 in (10.9), so

$$\nu = \frac{3}{2}\,. \tag{10.12}$$

Bessel functions of half-integer order are the spherical Bessel functions which can be expressed in terms of trigonometric functions. The solution (10.8) now reduces to

$$\delta\varphi_{\boldsymbol{k}}(t) = A_{\boldsymbol{k}}w_k(t) + B_{\boldsymbol{k}}w_k^*(t)\,, \tag{10.13}$$

where the constants $A_{\boldsymbol{k}}, B_{\boldsymbol{k}}$ have been redefined to absorb some numerical constants, compared to (10.8), and

$$w_k(t) = \left(i + \frac{k}{aH}\right)\exp\left(\frac{ik}{aH}\right)\ . \tag{10.14}$$

Well before horizon exit, $k \gg aH$, the argument of the exponent is large, and the solution oscillates rapidly. After horizon exit, $k \ll aH$, the solution stops oscillating and approaches the constant value $i(A_{\boldsymbol{k}} - B_{\boldsymbol{k}})$. (This fits in with our observation in chapter 9 that the scalar metric perturbation and the density become constant for $k \ll aH$.)

[1] One such gauge is the *spatially flat gauge*, where the scalar perturbations are chosen such that constant time slices have Euclidean geometry. There are still perturbations in the *spacetime* curvature, and they show up in the g_{0i} components of the metric.

10.1.3 The comoving curvature perturbation

We now have the solution for the field perturbation – or, more precisely, *a* field perturbation, that is to say, the field perturbation in a particular gauge. (The field perturbation is not a gauge-invariant quantity.) How is this field perturbation related to quantities in the longitudinal gauge we have used earlier? The price to pay for simplifying the equations of motion by judicious choice of gauge is that we have to deal with quantities in different gauges. A clean way to solve the problem is to use quantities that are gauge-invariant, that is to say, the same in every gauge. A central such quantity is the *the comoving curvature perturbation* $\mathcal{R}$. We won't go into the definition of this quantity: for us it is sufficient to know that its value is the same in all gauges. So if we calculate $\mathcal{R}$ in terms of $\delta\varphi$ in the gauge above, we can use the resulting value of $\mathcal{R}$ in any other gauge. The gauge invariant quantity is a "bridge" from one gauge to another, if you will.

In the gauge we used above, the comoving curvature perturbation is

$$\mathcal{R}_{\boldsymbol{k}} = -H\frac{\delta\varphi_{\boldsymbol{k}}}{\dot{\varphi}} \ . \tag{10.15}$$

So, we should calculate the inflaton field perturbation some time after horizon exit, when it has settled to a constant value, calculate $\mathcal{R}$ with (10.15). This is then a quantity which is gauge-independent and conserved outside the horizon, and we can calculate things like the density contrast δ from it (we will discuss this in the next chapter).

The pieces that we are missing are the constants of integration in (10.13), i.e. the initial conditions for the perturbation.

10.2 The generation of perturbations

It may sound somewhat odd to discuss the generation of perturbations. This implies that we consider the state of a system which is homogeneous and isotropic at some initial time, but where the behaviour is nevertheless different at different positions at a later time. This may seem impossible, because then we would have to a have a rule that would say where the perturbations are going to be, which would distinguish one position from another. Therefore it would seem that perturbations have to be given as initial conditions, and cannot be calculated from first principles. In a deterministic theory, this is true. However, quantum theory offers a way out of this impasse. It is is indeterministic, and there is no rule that will tell what the outcome of a quantum process will be, only probabilities of various outcomes (i.e. statistical distributions) are calculable. To discuss quantum behaviour of the inflaton field, we need to use quantum field theory in an inflating FRW universe. To warm up we first consider quantum field theory of a scalar field in Minkowski space.

10.2.1 Vacuum fluctuations in Minkowski space

The field equation for a massive free (i.e. $V(\varphi) = \frac{1}{2}m^2\varphi^2$) real scalar field in Minkowski space is

$$\ddot{\varphi} - \nabla^2\varphi + m^2\varphi = 0\,, \tag{10.16}$$

or

$$\ddot{\varphi}_{\boldsymbol{k}} + E_k^2\varphi_{\boldsymbol{k}} = 0\,, \tag{10.17}$$

where $E_k^2 = k^2 + m^2$, for Fourier components. We recognise (10.17) as the equation for a harmonic oscillator. Thus each Fourier component of the field behaves as an independent harmonic oscillator.

In the quantum mechanical treatment of the harmonic oscillator one introduces the creation and annihilation operators, which raise and lower the energy state of the system. We can do the same here.

Now we have a different pair of creation and annihilation operators $\hat{a}^\dagger_{\boldsymbol{k}}$, $\hat{a}_{\boldsymbol{k}}$ for each Fourier mode $\boldsymbol{k}$. We denote the ground state of the system by $|0\rangle$, and call it the *vacuum*. As discussed earlier, *particles* are quanta of the oscillations of the field. The vacuum is a state with no particles. Operating on the vacuum with the creation operator $\hat{a}^\dagger_{\boldsymbol{k}}$, we add one quantum with momentum $\boldsymbol{k}$ and energy E_k to the system, i.e. we create one particle. We denote this state with one particle with momentum is $\boldsymbol{k}$ by $|1_{\boldsymbol{k}}\rangle$. Thus

$$\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle = |1_{\boldsymbol{k}}\rangle \ , \tag{10.18}$$

and the state is normalised as $\langle 1_{\boldsymbol{k}}|1_{\boldsymbol{k}'}\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'}$. This particle has a well-defined momentum $\boldsymbol{k}$, and therefore it is completely unlocalised, as dictated by the Heisenberg uncertainty principle. The annihilation operator acting on the vacuum gives zero, i.e. not the vacuum state but the zero element of Hilbert space (the space of all quantum states),

$$\hat{a}_{\boldsymbol{k}}|0\rangle = 0 \ . \tag{10.19}$$

We denote the hermitian conjugate of the vacuum state by $\langle 0|$. Thus

$$\langle 0|\hat{a}_{\boldsymbol{k}} = \langle 1_{\boldsymbol{k}}| \qquad \text{and} \qquad \langle 0|\hat{a}^\dagger_{\boldsymbol{k}} = 0\,. \tag{10.20}$$

The commutation relations of the creation and annihilation operators are

$$[\hat{a}^\dagger_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}] = [\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}] = 0\,, \qquad [\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}] = \delta_{\boldsymbol{k}\boldsymbol{k}'} \ . \tag{10.21}$$

When going from classical physics to quantum physics, classical observables are replaced by operators. We can then calculate expectation values for these observables using the operators. Here the classical observable

$$\varphi(t, \boldsymbol{x}) = \sum \varphi_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{10.22}$$

is replaced by the *field operator*

$$\hat{\varphi}(t, \boldsymbol{x}) = \sum \hat{\varphi}_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{10.23}$$

where[2]

$$\hat{\varphi}_{\boldsymbol{k}}(t) = w_k(t)\hat{a}_{\boldsymbol{k}} + w_k^*(t)\hat{a}^\dagger_{-\boldsymbol{k}} \tag{10.24}$$

and

$$w_k(t) = L^{-3/2}\frac{1}{\sqrt{2E_k}}e^{-iE_k t} \tag{10.25}$$

is the mode function, a solution of the field equation (10.17). (The normalisation has been fixed to get the right commutation relations, (10.27).) We are using the

[2]We skip the detailed derivation of the field operator, which belongs to a course of quantum field theory. See e.g. Peskin & Schroeder, section 2.3 (note the different normalisations of operators and states, related to doing Fourier integrals rather than sums, and considerations of Lorentz invariance).

Heisenberg picture, i.e. we have time-dependent operators and the quantum states are time-independent. Note that since the operator $\hat{\varphi}(t, \boldsymbol{x})$ is Hermitian (corresponding to a real field), $\hat{\varphi}(t, \boldsymbol{x})^\dagger = \hat{\varphi}(t, \boldsymbol{x})$, the corresponding Fourier components satisfy $\hat{\varphi}_{\boldsymbol{k}}(t)^\dagger = \hat{\varphi}_{-\boldsymbol{k}}(t)$. So the Fourier component operators are not Hermitian.

In quantum mechanics, we have two conjugate variables, position and momentum. In quantum field theory, we have the field and the corresponding canonical momentum, which is in this case just given by the time derivative of the field. Combining (10.24) and (10.25), we have

$$\dot{\hat{\varphi}}_{\boldsymbol{k}}(t) = -iE_k \left(w_k(t)\hat{a}_{\boldsymbol{k}} - w_k^*(t)\hat{a}^\dagger_{-\boldsymbol{k}} \right) . \tag{10.26}$$

We can now calculate the commutator between the field operator and the corresponding velocity operator. A straightforward calculation with the rules (10.21) gives

$$[\hat{\varphi}_{\boldsymbol{k}}(t), \dot{\hat{\varphi}}_{\boldsymbol{k}'}(t)] = iL^{-3}\delta_{\boldsymbol{k},-\boldsymbol{k}'} . \tag{10.27}$$

(**Exercise:** Show that demanding the canonical commutation relation (10.27) fixes the normalisation to be the one given in (10.25).) Recall that the Lagrangean density of a scalar field is (in Minkowski space)

$$\hat{\mathcal{L}} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\hat{\varphi}\,\partial_\nu\hat{\varphi} - V(\hat{\varphi}) , \tag{10.28}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ as always. The corresponding Hamiltonian density is

$$\hat{\mathcal{H}} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\hat{\varphi}\,\partial_\nu\hat{\varphi} + V(\hat{\varphi}) , \tag{10.29}$$

and the Hamiltonian is the spatial integral of the Hamiltonian density,

$$\hat{H} = \int \mathrm{d}^3x\hat{\mathcal{H}} . \tag{10.30}$$

(Note that the Lagrangean density corresponds to the pressure of the scalar field, and the Hamiltonian density corresponds to the energy density.) Since the Hamiltonian depends on the field velocity operator, it does not commute with the field operator,

$$[\hat{H}, \hat{\varphi}] \neq 0 . \tag{10.31}$$

As a result, the Hamiltonian and the field operator do not share a complete set of eigenstates. So, in general an eigenstate of the Hamiltonian is not an eigenstate of the field operator. Eigenstates of the Hamiltonian operator are the energy eigenstates, and the state with the smallest energy is called the vacuum state. Since the vacuum is not an eigenstate of the field operator, the eigenvalues of the field operator are not well defined, instead we have only a distribution of values. In other words, the scalar field has *vacuum fluctuations*. It can be shown that these fluctuations are Gaussian (we skip the proof). This means that they are completely characterised by the power spectrum, as discussed in chapter 9.

It is straightforward to calculate the power spectrum, defined as

$$\mathcal{P}_\varphi(k) = L^3\frac{k^3}{2\pi^2}\langle|\varphi_{\boldsymbol{k}}|^2\rangle . \tag{10.32}$$

Recall that the power spectrum is related to the variance of the field as (note that $\langle\hat{\varphi}\rangle = 0$)

$$\langle\hat{\varphi}(\boldsymbol{x})^2\rangle = \int_0^\infty \frac{dk}{k}\mathcal{P}_\varphi(k) \ . \tag{10.33}$$

For the vacuum state $|0\rangle$, the expectation value of $|\varphi_{\boldsymbol{k}}|^2$ is

$$\begin{aligned}\langle 0|\hat{\varphi}_{\boldsymbol{k}}\hat{\varphi}^\dagger_{\boldsymbol{k}}|0\rangle \ &= \\ &|w_k|^2\langle 0|\hat{a}_{\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle + w_k^2\langle 0|\hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}|0\rangle + (w_k^*)^2\langle 0|\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}^\dagger_{\boldsymbol{k}}|0\rangle + |w_k|^2\langle 0|\hat{a}^\dagger_{-\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}|0\rangle \\ &= \ |w_k|^2\langle 1_{\boldsymbol{k}}|1_{\boldsymbol{k}}\rangle \ = \ |w_k|^2\end{aligned} \tag{10.34}$$

since all but the first term give 0, and the states are normalised so that $\langle 1_{\boldsymbol{k}}|1_{\boldsymbol{k}'}\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'}$. Therefore the power spectrum is

$$\mathcal{P}_\varphi(k) = L^3\frac{k^3}{2\pi^2}|w_k|^2 \ . \tag{10.35}$$

From (10.25) we have $|w_k|^2 = 1/(2L^3E_k)$, so we get the final result

$$\mathcal{P}_\varphi(k) = \frac{k^3}{4\pi^2 E_k} \ . \tag{10.36}$$

In the case of inflation, the mode functions are different because space is expanding, but the reasoning is the same.

10.2.2 Vacuum fluctuations during inflation

During inflation the field equation for inflaton perturbations is, from (10.6),

$$\delta\ddot{\varphi}_{\boldsymbol{k}} + 3H\delta\dot{\varphi}_{\boldsymbol{k}} + \left[\left(\frac{k}{a}\right)^2 + m^2(\bar{\varphi})\right]\delta\varphi_{\boldsymbol{k}} = 0 \ . \tag{10.37}$$

In inflation, the background field is treated classically, and only the perturbations around the mean value of the field are quantised. In fact, if we were to do the calculation in a gauge-independent manner, we would see that the variables which are quantised are a linear combination of the scalar field perturbations and metric perturbations. Thus in inflation, part of the spacetime metric is quantised. Inflation may thus be called the first quantum gravity scenario which has been confronted with observations – with great success. However, just like the background scalar field, the background metric is not quantised. How to quantise the metric in general, and not just small perturbations, remains one of the most studied and most difficult questions in physics. In this course, we just treat the field perturbation during inflation the same way that we treated the field in Minkowski space. That is, the Fourier modes of the field perturbation are written as

$$\delta\hat{\varphi}_{\boldsymbol{k}}(t) = w_k(t)\hat{a}_{\boldsymbol{k}} + w_k^*(t)\hat{a}^\dagger_{-\boldsymbol{k}} \ , \tag{10.38}$$

where the mode function $w_k(t)$ satisfies the classical equation of motion (10.6), with the normalisation fixed by the canonical commutation relation,

$$[\delta\hat{\varphi}_{\boldsymbol{k}}(t), \delta\dot{\hat{\varphi}}_{\boldsymbol{k}'}(t)] = i(aL)^{-3}\delta_{\boldsymbol{k},-\boldsymbol{k}'} \ , \tag{10.39}$$

where the only difference from the Minkowski space commutator (10.27) is the presence of a^{-3} on the right-hand side.

Taking the solution of (10.6) given in section 10.1.2, under the approximations $H =$ const. and $\frac{m^2}{H^2} = 3\eta \approx 0$ and fixing the normalisation with (10.39), we get the solution

$$w_k(t) = L^{-3/2}\frac{H}{\sqrt{2k^3}}\left(i + \frac{k}{aH}\right)\exp\left(\frac{ik}{aH}\right) , \tag{10.40}$$

where the time-dependence is $a(t) \propto e^{Ht}$.

When the scale k is well inside the horizon, $k \gg aH$, $\delta\varphi_{\boldsymbol{k}}(t)$ oscillates rapidly compared to the Hubble time H^{-1}. If we consider distance and time scales much smaller than the Hubble scale, spacetime curvature does not matter and things should behave like in Minkowski space. Considering (10.40) in this limit, one finds (**exercise**) that $w_k(t)$ indeed becomes (up to a slowly varying phase), equal to the Minkowski space mode function (10.25), with the lengths scaled by a. (The prefactor in (10.40) was chosen so that the normalisations would agree.) Therefore the mode function $w_k(t)$ of (10.40) tells us how the perturbation behaves as it approaches and exits the horizon.

The calculation of the power spectrum of inflaton fluctuations is the same as in Minkowski space, with the same result,

$$\mathcal{P}_{\delta\varphi}(k) = L^3\frac{k^3}{2\pi^2}|w_k|^2 . \tag{10.41}$$

Well before horizon exit, $k \gg aH$, and on timescales $\ll H^{-1}$, the field operator $\delta\hat{\varphi}_{\boldsymbol{k}}(t)$ agrees with the Minkowski space field operator and we have the same kind of initial $\delta\varphi$ vacuum fluctuations as in Minkowski space. However, the time evolution of the perturbations is different. Well after horizon exit, $k \ll aH$, the mode function approaches a constant

$$w_k(t) \;\to\; L^{-3/2}\frac{iH}{\sqrt{2k^3}} , \tag{10.42}$$

so the vacuum fluctuations "freeze" and the power spectrum acquires the constant value

$$\mathcal{P}_{\delta\varphi}(k) \;=\; L^3\frac{k^3}{2\pi^2}|w_k|^2 \;=\; \left(\frac{H}{2\pi}\right)^2 . \tag{10.43}$$

We have calculated the power spectrum of the inflaton field perturbations by using the quantum mechanical expectation value of the square of the field perturbation. We now identify this with the expectation value of a probability distribution of a classical variable, i.e. we assume that the quantum mechanical fluctuations become classical. Some part of this process is understood (it can be shown that the quantum mechanical expectation values become equal to those of a classical stochastic distribution, or "squeezed"), but the emergence of (at least the appearance of) classical reality from a quantum system remains an unsolved problem. In particle physics appeal is often made to the Copenhagen interpretation according to which states become classical when they are measured, but for cosmology this is inadequate. We simply assume that we can replace an expectation value of a quantum state with the ensemble average of a classical distribution.

For our purposes, quantum mechanics generates the initial perturbations and solves the problem of how perturbations can emerge from a state which is homogeneous and isotropic. As a remnant of the indeterministic origin of the perturbations, we cannot predict the specific member of the ensemble which is realised in the universe, we can only calculate the statistical distribution of perturbations. As noted, this distribution is Gaussian, so all Fourier modes $\delta\varphi_{\boldsymbol{k}}$ acquire their values as independent random variables (except for the reality condition $\delta\varphi_{-\boldsymbol{k}} = \delta\varphi^*_{\boldsymbol{k}}$) with a Gaussian probability distribution.

The result (10.43) was obtained treating H as a constant. However, H does change, albeit slowly, during inflation. The main purpose of our discussion was to follow the inflaton perturbations through the horizon exit. After the perturbation is well outside the horizon, we switch to other variables, namely the curvature perturbation $\mathcal{R}_{\boldsymbol{k}}$ which remains constant outside the horizon even though H changes, unlike $\delta\varphi_{\boldsymbol{k}}$ (we see from (10.37) that $\delta\varphi_{\boldsymbol{k}}$ is not constant in general). To take into account evolution we use for each scale k the value of H which is representative for the evolution of that particular scale through the horizon. That is, we choose the value of H at horizon exit[3], so that $aH = k$. Thus the power spectrum is

$$\mathcal{P}_{\delta\varphi}(k) \;=\; L^3 \frac{k^3}{2\pi^2}|w_k|^2 \;=\; \left(\frac{H}{2\pi}\right)^2_{aH=k} , \tag{10.44}$$

where the subscript notation signifies that the value of H for each k is to be taken at horizon exit of that particular scale.

Since we have only one quantity which has fluctuations, the inflaton field, and the perturbations are treated in linear theory, the perturbations of any other quantity are related to the inflaton field fluctuation by linear and local equations. In other words, any perturbation quantity $g_{\boldsymbol{k}}$ depends only on the field perturbation $\delta\varphi_{\boldsymbol{k}}$ with the same wavenumber, $g_{\boldsymbol{k}}(t) = f_{g\varphi}(t,k)\delta\varphi_{\boldsymbol{k}}(t_k)$. Thus the statistics of the inflaton perturbations $\delta\varphi(\boldsymbol{x})$ are inherited by all other perturbations, and we have $\mathcal{P}_g(k) = f_{g\varphi}(t,k)^2\mathcal{P}_{\delta\varphi}(k)$. The function $f_{g\varphi}(t,k)$ (like the power spectrum $\mathcal{P}_{\delta\varphi}(k)$) can only depend on the magnitude k, not on the direction of $\boldsymbol{k}$, because the background is homogeneous and isotropic. So the distribution of the perturbations inherits the property of homogeneity and isotropy from the symmetry of the background on which they are created and evolve: perturbations generated by inflation are statistically homogeneous and isotropic.

In particular, for the comoving curvature perturbation we have, from (10.15),

$$\mathcal{R}_{\boldsymbol{k}} = -H\frac{\delta\varphi_{\boldsymbol{k}}}{\dot{\bar{\varphi}}} , \tag{10.45}$$

so we obtain

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\bar{\varphi}}}\right)^2 \mathcal{P}_{\delta\varphi}(k) = \left(\frac{H}{\dot{\bar{\varphi}}}\frac{H}{2\pi}\right)^2_{aH=k} . \tag{10.46}$$

This the main result for quantum fluctuations during inflation. The problem has now been completely reduced to the evolution of the background scalar field and the background Hubble parameter. We just need to specify the inflation potential and

[3]One can do a more precise calculation, where one takes into account the evolution of $H(t)$. The result is that one gets a correction to the amplitude of $\mathcal{P}_{\mathcal{R}}(k)$, which is first order in slow-roll parameters and a correction to its spectral index n which is second order in the slow-roll parameters. Note that H is assumed to be constant only for each k mode during the time it crosses the horizon. The equations of motion of the different modes are independent, so in principle H could be very different for modes that exit at very different times without violating our assumptions.

calculate how the background evolves, and plug it in (10.46) to get complete information about the perturbations. That, in turn, is the starting point for calculating structure formation and the CMB anisotropy. Turning this around, observations of large-scale structure and the CMB can be used obtain information about quantum processes in the primordial universe. Note that the power spectrum depends only on k. Statistical homogeneity and isotropy of the perturbations, inherited from the symmetry of the background, is a strong feature of inflation. (I use the word 'feature' rather than 'prediction', because it is possible to construct models where, for example, space expands anisotropically during inflation. However, that requires untypical assumptions, such as having a short period of inflation, so that the anisotropy is not washed away, or inflation driven by a vector field instead of a scalar field.)

10.3 The primordial spectrum in slow-roll inflation

So, inflation generates primordial perturbations $\mathcal{R}_{\boldsymbol{k}}$ with the power spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\varphi}}\frac{H}{2\pi}\right)^2_{aH=k}, \tag{10.47}$$

(In this section, we drop the overbar from the background values.) Let's now get back to the inflaton potential and the presentation of the dynamics of slow-roll inflation in terms of the two slow-roll variables. Applying the slow-roll equations

$$H^2 = \frac{V}{3M_{\text{Pl}}^2} \qquad \text{and} \qquad 3H\dot{\varphi} = -V'$$

(10.47) becomes

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{12\pi^2}\frac{1}{M_{\text{Pl}}^6}\frac{V^3}{V'^2} = \frac{1}{24\pi^2}\frac{1}{M_{\text{Pl}}^4}\frac{V}{\varepsilon}, \tag{10.48}$$

where ε is the slow-roll parameter.

According to observations of CMB and large-scale structure, the amplitude of the primordial power spectrum is

$$\mathcal{P}_{\mathcal{R}}(k)^{1/2} \approx 5 \times 10^{-5} \tag{10.49}$$

on cosmological scales. This gives a constraint on inflation

$$\left(\frac{V}{\varepsilon}\right)^{1/4} \approx 24^{1/4}\sqrt{\pi}\sqrt{5\times 10^{-5}}M_{\text{Pl}} \approx 0.028 M_{\text{Pl}} = 6.8\times 10^{16}\ \text{GeV}. \tag{10.50}$$

Since $\varepsilon \ll 1$, this implies an upper limit on the energy scale of inflation,

$$V^{1/4} < 0.028 M_{\text{Pl}}. \tag{10.51}$$

This puts a limit on the Hubble scale during inflation. From $H^2 = V/(3M_{\text{Pl}}^2)$, the constraint $V^{1/4} < 6.8 \times 10^{16}$ GeV translates into $H < 10^{15}$ GeV, or in terms of length, $H^{-1} > 10^{-31}$ m.

Since during slow-roll inflation V and V' change slowly while a wide range of scales k exit the horizon, we expect $\mathcal{P}_{\mathcal{R}}(k)$ to be a slowly varying function of k. We describe this small variation with the *spectral index* n of the primordial spectrum, defined as[4]

$$n(k) - 1 \equiv \frac{d \ln \mathcal{P}_\mathcal{R}}{d \ln k} . \tag{10.52}$$

If the spectral index is independent of k, we say that the spectrum is *scale-free.* In this case the primordial spectrum is a *power law*

$$\mathcal{P}_\mathcal{R}(k) = A^2 \left(\frac{k}{k_p}\right)^{n-1} , \tag{10.53}$$

where the"pivot scale" k_p is some chosen reference scale and A is the amplitude at this pivot scale.

If the power spectrum is constant,

$$\mathcal{P}_\mathcal{R}(k) = \text{const.} , \tag{10.54}$$

corresponding to $n = 1$, we say that the spectrum is *scale-invariant* (which is a special case of a scale-free spectrum.). A scale-invariant spectrum is also called the *Harrison–Zel'dovich* spectrum.

If $n \neq 1$, the spectrum is called *tilted.* A tilted spectrum is called *red* if $n < 1$ (more power on large scales) and *blue* if $n > 1$ (more power on small scales). If $dn/dk \neq 0$, it is said that there is a *running spectral index.*

Using (10.48) and (10.52), we can calculate the spectral index for slow-roll inflation. Since $\mathcal{P}_\mathcal{R}(k)$ is evaluated from (10.48) when $k = aH$, we have

$$\frac{d \ln k}{dt} = \frac{d \ln(aH)}{dt} = \frac{\dot{a}}{a} + \frac{\dot{H}}{H} = (1 - \varepsilon)H ,$$

where we used the fact that in the slow-roll approximation $\dot{H} = -\varepsilon H^2$ in the last step. Thus

$$\frac{d}{d \ln k} = \frac{1}{1-\varepsilon}\frac{1}{H}\frac{d}{dt} = \frac{1}{1-\varepsilon}\frac{\dot{\varphi}}{H}\frac{d}{d\varphi} = -\frac{M_{\text{Pl}}^2}{1-\varepsilon}\frac{V'}{V}\frac{d}{d\varphi} \approx -M_{\text{Pl}}^2 \frac{V'}{V}\frac{d}{d\varphi} . \tag{10.55}$$

Let us first calculate the scale dependence of the slow-roll parameters:

$$\frac{d\varepsilon}{d \ln k} = -M_{\text{Pl}}^2 \frac{V'}{V}\frac{d}{d\varphi}\left[\frac{M_{\text{Pl}}^2}{2}\left(\frac{V'}{V}\right)^2\right] = M_{\text{Pl}}^4\left[\left(\frac{V'}{V}\right)^4 - \left(\frac{V'}{V}\right)^2 \frac{V''}{V}\right] = 4\varepsilon^2 - 2\varepsilon\eta \tag{10.56}$$

and, in a similar manner (**exercise**),

$$\frac{d\eta}{d \ln k} = \ldots = 2\varepsilon\eta - \xi , \tag{10.57}$$

where we have defined a third slow-roll parameter

$$\xi \equiv M_{\text{Pl}}^4 \frac{V'}{V^2} V''' . \tag{10.58}$$

[4]The -1 is in the definition for historical reasons, related to other ways of defining the power spectrum of perturbations.

The parameter ξ is typically second-order small in the sense that $\sqrt{|\xi|}$ is of the same order of magnitude as ε and η. (Therefore it is sometimes written as ξ^2, although this can be misleading, as it does not have to be positive.)

We can now calculate the spectral index:

$$\begin{aligned} n-1 &= \frac{1}{\mathcal{P}_\mathcal{R}}\frac{d\mathcal{P}_\mathcal{R}}{d\ln k} = \frac{\varepsilon}{V}\frac{d}{d\ln k}\left(\frac{V}{\varepsilon}\right) = \frac{1}{V}\frac{dV}{d\ln k} - \frac{1}{\varepsilon}\frac{d\varepsilon}{d\ln k} \\ &= -M_{\mathrm{Pl}}^2\frac{V'}{V}\cdot\frac{1}{V}\frac{dV}{d\varphi} - 4\varepsilon + 2\eta \;=\; -6\varepsilon + 2\eta\ . \end{aligned} \tag{10.59}$$

Slow-roll requires $\varepsilon \ll 1$ and $|\eta| \ll 1$, so the spectrum is predicted to be close to scale invariant. This agrees well with observations. Note how, as in the case of dark matter, things fall into place automatically. In order to have negative pressure, a scalar field has to roll slowly. Once the background evolution is slowly rolling, the perturbations are close to scale-invariant, without needing to add new ingredients or tune anything.

Assuming that at late times the universe is described by the ΛCDM model, the current constraint on the spectral index using the minimal dataset of the Planck satellite [1]

$$n = 0.9655 \pm 0.0062\ . \tag{10.60}$$

Adding other cosmological data, the strongest constraint is $n = 0.9667 \pm 0.0040$. Note that while the statistics improves as new data is added, the systematic reliability of the values quoted can decrease, because the quoted number is now dependent on the assumption that systematic issues in all datasets are completely understood. The value is also model-dependent, and with a different cosmological model (different dark energy, isocurvature perturbations –to which we come in the next chapter–, topological defects and so on), the preferred value of the spectral index can change. However, in all but the most exotic models it remains close to scale-invariant.

From the results of the running of ε and η, we obtain the running of the spectral index:

$$\frac{dn}{d\ln k} = 16\varepsilon\eta - 24\varepsilon^2 - 2\xi\ . \tag{10.61}$$

The running is second order in slow-roll parameters, so it's expected to be even smaller than the deviation from scale invariance. The observational range is, according to the minimal Planck dataset, [1]

$$\frac{dn}{d\ln k} = -0.0126^{+0.0098}_{-0.0087}\ , \tag{10.62}$$

and the addition of other datasets improves this to $\frac{dn}{d\ln k} = -0.0065 \pm 0.0076$. Some inflation models have $|n-1|$ and $|dn/d\ln k|$ larger than this, while others do not. Observations have ruled out some inflation models, while a zoo of dozens of viable models remains [2].

CMB experiments have measured the CMB temperature anisotropy over a range $\Delta\ln k \approx 8$. On scales smaller than those that have been probed, the CMB anisotropy is expected to be negligible (see chapter 12 for the reason why!), so there's nothing more to find in the CMB temperature anisotropies. However, it is possible to probe these smaller scales by observations of large-scale structure. Recall that for high energy-scale inflation, the number of e-folds until the end of inflation when the largest observable modes are generated is about 60, so we are only seeing a small part of inflation.

The above results do not yet allow an independent determination of the two slow-roll parameters ε and η. However, it turns out that the spectral index of *tensor perturbations* produced by inflation is independent of η (it is -2ε). So if tensor perturbations are detected (they have a definite signature on the CMB) and their spectrum is measured, we can get both ε and η. The amplitude of the tensor perturbations also depends directly on the Hubble parameter on inflation, so it will provide a measurement of the energy scale of inflation. Typically, large-field inflation models produce tensor perturbations with much larger amplitude than small-field inflation models. In the small-field case they may be too small to be detectable in the near future. It is possible to calculate the spectrum of gravity waves the same way as we did for the scalar perturbations (the calculation is in fact simpler in the sense that the gravity waves do not have a mass term, and they are gauge-invariant, unlike the scalar field perturbations).

Example: Consider the simple inflation model

$$V(\varphi) = \frac{1}{2}m^2\varphi^2\,. \tag{10.63}$$

In chapter 8 we already calculated the slow-roll parameters for this model:

$$\varepsilon = \eta = 2\frac{M_{\rm Pl}^2}{\varphi^2} \tag{10.64}$$

and we immediately see that $\xi = 0$. We thus have

$$\mathcal{P}_\mathcal{R} = \frac{1}{96\pi^2}\frac{m^2}{M_{\rm Pl}^2}\left(\frac{\varphi}{M_{\rm Pl}}\right)^4 \tag{10.65}$$

$$n = 1 - 6\varepsilon + 2\eta = 1 - 8\left(\frac{M_{\rm Pl}}{\varphi}\right)^2 \tag{10.66}$$

$$\frac{dn}{d\ln k} = 16\varepsilon\eta - 24\varepsilon^2 - 2\xi = -32\left(\frac{M_{\rm Pl}}{\varphi}\right)^4\,. \tag{10.67}$$

To get the numbers out, we need the values of φ when the relevant cosmological scales left the horizon. We know that the number of inflation e-foldings after that should be about $N \approx 50\ldots60$, depending on the preheating history. We have

$$N(\varphi) = \frac{1}{M_{\rm Pl}^2}\int_{\varphi_{\rm end}}^{\varphi}\frac{V}{V'}d\varphi = \frac{1}{M_{\rm Pl}^2}\int\frac{\varphi}{2}d\varphi = \frac{1}{4M_{\rm Pl}^2}\left(\varphi^2 - {\varphi_{\rm end}}^2\right)\,, \tag{10.68}$$

and we estimate $\varphi_{\rm end}$ from $\varepsilon(\varphi_{\rm end}) = 2M_{\rm Pl}^2/{\varphi_{\rm end}}^2 = 1 \quad\Rightarrow\quad \varphi_{\rm end} = \sqrt{2}M_{\rm Pl}$ to get

$$\varphi^2 = {\varphi_{\rm end}}^2 + 4M_{\rm Pl}^2N = 2M_{\rm Pl}^2 + 4M_{\rm Pl}^2N \approx 4M_{\rm Pl}^2N\,. \tag{10.69}$$

Thus

$$\left(\frac{M_{\rm Pl}}{\varphi}\right)^2 = \frac{1}{4N}\,, \tag{10.70}$$

so we get

$$\begin{aligned}\mathcal{P}_\mathcal{R} &= \frac{N^2}{6\pi^2}\frac{m^2}{M_{\rm Pl}^2} = \frac{1250}{3\pi^2}\frac{m^2}{M_{\rm Pl}^2} \qquad (10.71)\\ n &= 1-\frac{2}{N} = 0.96 \\ \frac{dn}{d\ln k} &= -\frac{2}{N^2} = -0.0008\ , \qquad (10.72)\end{aligned}$$

where we havee for n and $\frac{dn}{d\ln k}$ input $N = 50$. For $\mathcal{P}_\mathcal{R}$ we have, according to (10.49) $\mathcal{P}_\mathcal{R} = 25\times 10^{-10}$, which gives

$$m \approx \frac{9}{N}10^{14}\ \text{GeV} \approx 2\times 10^{13}\ \text{GeV} \approx 8\times 10^{-6}M_{\rm Pl}\ , \qquad (10.73)$$

for $N = 50$. We get $V^{1/4} = (2Nm^2M_{\rm Pl}^2)^{1/4} \approx 2\times 10^{16}$ GeV as the energy scale for the period when the perturbations seen in the CMB were generated. Potential energy at the end of inflation is

$$V_{\rm end}^{1/4} = \left(\frac{1}{2}m^2{\varphi_{\rm end}}^2\right)^{1/4} = \sqrt{\frac{m}{M_{\rm Pl}}}M_{\rm Pl} \approx 3\times 10^{-3}M_{\rm Pl} \approx 7\times 10^{15}\ \text{GeV}\ . \qquad (10.74)$$

Because of the high energy scale, the amplitude of tensor perturbations, as quantified by *the tensor-to-scalar* ratio r is significant, $r \approx 0.1$. There was an announcement in March 2014 by the BICEP2 telescope team that inflationary gravity waves would have been detected with r in this range, but this turned out to be premature. There is no evidence for primordial gravitational waves, and the current upper limit from combined Planck and BICEP2/Keck data is $r < 0.07$ [3]. Therefore, the simple $m^2\varphi^2$ model is now disfavoured, although it fitted the observations fine until the Planck data.

Exercise: It can be shown that the power spectrum of gravity waves produced by inflation is

$$\mathcal{P}_t(k) = \frac{8}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2_{aH=k}\ .$$

(This power spectrum is related to the metric perturbation h_{ij}; we skip the definition.) Find the tensor-to-scalar ratio

$$r \equiv \frac{\mathcal{P}_t(k)}{\mathcal{P}_\mathcal{R}(k)}$$

and the tensor spectral index

$$n_t \equiv \frac{d\ln\mathcal{P}_t}{d\ln k}$$

in terms of the slow-roll parameters to first order.

Exercise: From the limit $r < 0.07$, calculate the resulting limit on the energy scale of inflation. Using that, find the maximum amount by which the scale factor can have expanded from reheating until today, assuming there are only Standard Model degrees of freedom.

References

[1] Planck Collaboration, [arXiv:1502.01589 [astro-ph.CO]]

[2] J. Martin, C. Ringeval and V. Vennin, Phys. Dark Univ. (2014) [arXiv:1303.3787 [astro-ph.CO]]

[3] K. Array *et al.* [BICEP2 s Collaboration], [arXiv:1510.09217 [astro-ph.CO]]