COSMOLOGY I Homework 4 Fall 2015

Due on September 28 by 14.15.

1. **Matter–radiation equality.** The present energy density of matter is $\rho_{m0} = \Omega_{m0}\rho_c$ and the present energy density of radiation is $\rho_{r0} = \rho_{\gamma 0} + \rho_{\nu 0}$, where $\rho_{\gamma 0} = AT_0^4$ is the contribution of the microwave background ($T_0 = 2.725$K) and $\rho_{\nu 0} = (21/8)AT_{\nu 0}^4$ is the contribution of the neutrino background (we assume neutrinos are massless). Here $A = \pi^2/15$ and $T_{\nu 0} = (4/11)^{1/3}T_0$.

   a) What was the age of the universe $t_{\rm eq}$ when $\rho_m = \rho_r$? (Note that at these early times—but not today—you can ignore the curvature and vacuum terms in the Friedmann equation; you don't need to make other assumptions about the values of Ω_0 or $\Omega_{\Lambda 0}$, since the answer does not depend on them.) Give the numerical value (in years) for the cases $\Omega_{m0} = 0.1$, 0.3 and 1.0, assuming $H_0 = 70$ km/s/Mpc.

   b) What is the temperature $T_{\rm eq} \equiv T(t_{\rm eq})$? Give the numerical value in the three different cases.

2. **Thermal distributions in the relativistic limit.** Derive the following equations in the limit $T \gg m, T \gg |\mu|$.

$$n = \begin{cases} \frac{3}{4\pi^2}\zeta(3)gT^3 & \text{fermions} \\ \frac{1}{\pi^2}\zeta(3)gT^3 & \text{bosons} \end{cases}$$

$$\rho = \begin{cases} \frac{7}{8}\frac{\pi^2}{30}gT^4 & \text{fermions} \\ \frac{\pi^2}{30}gT^4 & \text{bosons} \end{cases}$$

$$p = \frac{1}{3}\rho$$

$$\langle E \rangle = \begin{cases} \frac{7\pi^4}{180\zeta(3)}T & \text{fermions} \\ \frac{\pi^4}{30\zeta(3)}T & \text{bosons .} \end{cases}$$

   Bonus problem. This problem is worth one extra point. Derive the following equation for fermions in the limit $T \gg m$.

$$n - \bar{n} = \frac{gT^3}{6\pi^2}\left[\pi^2\left(\frac{\mu}{T}\right) + \left(\frac{\mu}{T}\right)^3\right] .$$

3. **Thermal distributions in the non-relativistic limit.** Derive the following equations for non-relativistic Maxwell-Boltzmann statistics ($T \ll m$ and $T \ll |m - \mu|$).

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{m-\mu}{T}}$$

$$\rho = n\left(m + \frac{3T}{2}\right)$$

$$p = nT$$

$$\langle E \rangle = m + \frac{3T}{2}$$

$$n - \bar{n} = 2g\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m}{T}} \sinh\frac{\mu}{T} .$$