

Due on Monday November 16 by 12.15.

1. **Inflaton perturbations.** The field equation for the inflaton perturbation is (in the spatially flat gauge, neglecting slow-roll-suppressed parameters)

$$\delta\ddot{\varphi}_{\mathbf{k}} + 3H\delta\dot{\varphi}_{\mathbf{k}} + \left[\left(\frac{k}{a}\right)^2 + V''(\bar{\varphi}) \right] \delta\varphi_{\mathbf{k}} = 0 .$$

Make the approximations $H = \text{const.}$ ($\Rightarrow a = e^{Ht}$) and $V''(\bar{\varphi}) = 0$ (consistent with neglecting terms that are slow-roll-suppressed).

- a) Show that

$$\delta\varphi_{\mathbf{k}}(t) = C(\mathbf{k})\eta^{\frac{3}{2}}H_{\frac{3}{2}}(k\eta) ,$$

where

$$H_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}e^{-ix}\left(1 + \frac{1}{ix}\right)$$

is a Hankel function with $\eta \equiv -H^{-1}e^{-Ht}$, is a solution.

- b) Show that η is the conformal time, and that the solution agrees with the one given in the lecture notes,

$$w_{\mathbf{k}}(t) = \left(i + \frac{k}{aH}\right) \exp\left(\frac{ik}{aH}\right) .$$

- c) What happens to the perturbation (its amplitude, distance scale and oscillation time scale) as inflation proceeds?

2. **Inflaton perturbations as vacuum fluctuations.**

- a) Starting from the canonical commutation relation for the quantised perturbation, show that with the correct normalisation factor, $\delta\varphi_{\mathbf{k}}(t)$ of the previous problem is

$$\delta\varphi_{\mathbf{k}}(t) = L^{-3/2} \frac{H}{\sqrt{2k^3}} \left(i + \frac{k}{aH}\right) \exp\left(\frac{ik}{aH}\right) .$$

- b) Show that on time and length scales much smaller than the Hubble scale, $t \ll H^{-1}$, $k \gg aH$, this becomes (up to a slowly varying phase factor) the Minkowski space mode function

$$w_{\mathbf{k}}(t) = (aL)^{-3/2} \frac{1}{\sqrt{2E_k}} e^{-iE_k t} .$$

- c) What is E_k ?

3. **Running of the spectral index.** Assuming slow-roll, calculate $d\varepsilon/d(\ln k)$, $d\eta/d(\ln k)$ and then $dn/d(\ln k)$ in terms of the slow-roll parameters to first non-trivial order.