COSMOLOGY II   Homework 10   Fall 2015

Due on Monday November 23 by 12.15.

1. **Tensor perturbations.** It can be shown that the power spectrum of gravity waves produced by inflation is
$$\mathcal{P}_t(k) = \frac{2}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2_{aH=k} .$$
(This power spectrum is related to the metric perturbation h_{ij}; we skip the definition.) Find the tensor-to-scalar ratio
$$r \equiv \frac{\mathcal{P}_t(k)}{\mathcal{P}_\mathcal{R}(k)}$$
and the tensor spectral index
$$n_t \equiv \frac{d\ln\mathcal{P}_t}{d\ln k}$$
in terms of the slow-roll parameters to first order.

2. **Energy scale of inflation.** The current upper limit from CMB data on the scalar-tensor ratio is $r < 0.07$.

   a) Calculate the resulting limit on the energy scale of inflation.

   b) Find the maximum amount by which the scale factor can have expanded from reheating until today, assuming there are only Standard Model degrees of freedom.

3. **Sound waves.** For short wavelength modes with $k \gg k_J$, density perturbations in the matter-dominated universe satisfy
$$\ddot{\delta}_{\boldsymbol{k}} + 2H\dot{\delta}_{\boldsymbol{k}} + c_s^2\frac{k^2}{a^2}\delta_{\boldsymbol{k}} = 0 .$$
Solve $\delta_{\boldsymbol{k}}(t)$, assuming that matter is the only component present and that $\Omega_m = 1$. How does the amplitude and frequency of the oscillations change with time and scale factor? (Conformal time $\eta = \int dt/a$ may be helpful.)

4. **CDM density perturbation in the radiation-dominated era.** Let us consider deeply sub-Hubble scales in the radiation-dominated era (and ignore decaying modes), so that the gravitational potential is given by eq. (11.40). Assume that perturbations are adiabatic (and remember throughout that we consider the limit $k \gg aH$).

   a) Show that the solution of eq. (11.41) is
$$\delta_{c\boldsymbol{k}} = \tilde{A}_{1\boldsymbol{k}} + \tilde{A}_{2\boldsymbol{k}}\ln y + \tilde{A}_{3\boldsymbol{k}}\int_y^\infty \frac{\mathrm{d}y'}{y'}\int_{y'}^\infty \mathrm{d}y''\frac{\cos y''}{y''} ,$$
where $y = k/(\sqrt{3}aH)$. What are the constants $\tilde{A}_{1\boldsymbol{k}}$ and $\tilde{A}_{2\boldsymbol{k}}$ in terms of $A_{1\boldsymbol{k}}$?

   b) Show that the last term is subdominant. (You can do this either by numerical plotting or analytically.)