4 Thermal history of the Early Universe

4.1 Relativistic thermodynamics

As we look out in space we can see the history of the universe unfolding in front of our telescopes. However, at redshift \( z = 1090 \) our line of sight hits the last scattering surface, from which the cosmic microwave background (CMB) radiation originates. This corresponds to \( t = 370000 \) years. Before that the universe was not transparent, so we cannot see further back in time, into the early universe. As explained in Sec. 3, we can ignore curvature and vacuum/dark energy in the early universe and concern ourselves only with radiation and matter. The isotropy of the CMB shows that matter was distributed homogeneously in the early universe, and the spectrum of the CMB shows that this matter, the “primordial soup” of particles, was in thermodynamic equilibrium. Therefore we can use thermodynamics to calculate the history of the early universe. As we shall see, this calculation leads to predictions (especially the BBN, big bang nucleosynthesis) testable by observation. We shall now discuss the thermodynamics of the primordial soup.

From elementary quantum mechanics we are familiar with the “particle in a box”. Let us consider a cubic box, whose edge is \( L \) (volume \( V = L^3 \)), with periodic boundary conditions. Solving the Schrödinger equation gives us the energy and momentum eigenstates, where the possible momentum values are

\[
\vec{p} = \frac{h}{L}(n_1 \hat{x} + n_2 \hat{y} + n_3 \hat{z}) \quad (n_i = 0, \pm 1, \pm 2, \ldots),
\]

where \( h \) is the Planck constant. (The wave function will have an integer number of wavelengths in each of the three directions.) The state density in momentum space (number of states / \( \Delta p_x \Delta p_y \Delta p_z \)) is thus

\[
\frac{L^3}{h^3} = \frac{V}{h^3},
\]

and the state density in the 6-dimensional phase space \( \{(\vec{x}, \vec{p})\} \) is \( 1/h^3 \). If the particle has \( g \) internal degrees of freedom (e.g., spin),

\[
\text{density of states} = \frac{g}{h^3} = \frac{g}{(2\pi)^3} \quad \left( h \equiv \frac{h}{2\pi} \equiv 1 \right).
\]

This result is true even for relativistic momenta. The state density in phase space is independent of the volume \( V \), so we can apply it for arbitrarily large systems (e.g., the universe).

For much of the early universe, we can ignore the interaction energies between the particles. Then the particle energy is (according to special relativity)

\[
E(\vec{p}) = \sqrt{p^2 + m^2},
\]

where \( p \equiv |\vec{p}| \) (not pressure!), and the states available for the particles are the free particle states discussed above.

Particles fall into two classes, fermions and bosons. Fermions obey the Pauli exclusion principle: no two fermions can be in the same state.

In thermodynamic equilibrium the distribution function, or the expectation value \( f \) of the occupation number of a state, depends only on the energy of the state. According to statistical physics, it is

\[
f(\vec{p}) = \frac{1}{e^{(E-\mu)/T} \pm 1}
\]

where + is for fermions and − is for bosons. (For fermions, where \( f \leq 1 \), \( f \) gives the probability that a state is occupied.) This equilibrium distribution has two parameters, the temperature \( T \),
and the chemical potential $\mu$. The temperature is related to the energy density in the system and the chemical potential is related to the number density $n$ of particles in the system. Note that, since we are using the relativistic formula for the particle energy $E$, which includes the mass $m$, it is also “included” in the chemical potential $\mu$. Thus in the nonrelativistic limit, both $E$ and $\mu$ differ from the corresponding quantities of nonrelativistic statistical physics by $m$, so that $E - \mu$ and the distribution functions remain the same.

If there is no conserved particle number in the system (e.g., a photon gas), then $\mu = 0$ in equilibrium.

The particle density in phase space is the density of states times their occupation number,

$$\frac{g}{(2\pi)^3} f(\vec{p}). \quad (6)$$

We get the particle density in (ordinary 3D) space by integrating over the momentum space. Thus we find the following quantities:

- number density $n = \frac{g}{(2\pi)^3} \int f(\vec{p}) d^3 p \quad (7)$
- energy density $\rho = \frac{g}{(2\pi)^3} \int E(\vec{p}) f(\vec{p}) d^3 p \quad (8)$
- pressure $p = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E} f(\vec{p}) d^3 p. \quad (9)$

Different particle species $i$ have different masses $m_i$; so the preceding is applied separately to each particle species. If particle species $i$ has the above distribution for some $\mu_i$ and $T_i$, we say the species is in kinetic equilibrium. If the system is in thermal equilibrium, all species have the same temperature, $T_i = T$. If the system is in chemical equilibrium (“chemistry” here refers to reactions where particles change into other species), the chemical potentials of different particle species are related according to the reaction formulas. For example, if we have a reaction

$$i + j \leftrightarrow k + l, \quad (10)$$

then

$$\mu_i + \mu_j = \mu_k + \mu_l. \quad (11)$$

Thus all chemical potentials can be expressed in terms of the chemical potentials of conserved quantities, e.g., the baryon number chemical potential, $\mu_B$. There are thus as many independent chemical potentials, as there are independent conserved particle numbers. For example, if the chemical potential of particle species $i$ is $\mu_i$, then the chemical potential of the corresponding antiparticle is $-\mu_i$. We can also have a situation that some reactions are in chemical equilibrium but others are not.

Thermodynamic equilibrium refers to having all these equilibria, but I will also use the term more loosely to refer to some subset of them.

As the universe expands, $T$ and $\mu$ change, so that energy continuity and particle number conservation are satisfied. In principle, an expanding universe is not in equilibrium. The expansion is however so slow, that the particle soup usually has time to settle close to local equilibrium. (And since the universe is homogeneous, the local values of thermodynamic quantities are also global values).

From the remaining numbers of fermions (electrons and nucleons) in the present universe, we can conclude that in the early universe we had $|\mu| \ll T$ when $T \gg m$. (We don’t know the chemical potential of neutrinos, but it is usually assumed to be small too). If the temperature is much greater than the mass of a particle, $T \gg m$, the ultrarelativistic limit, we can approximate $E = \sqrt{p^2 + m^2} \approx p$. 
For $|\mu| \ll T$ and $m \ll T$, we approximate $\mu = 0$ and $m = 0$ to get the following formulae

\begin{align*}
    n &= \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi p^2 \, dp}{e^p/T \pm 1} = \begin{cases} 
    \frac{3}{4\pi^2} \zeta(3) g T^3 & \text{fermions} \\
    \frac{1}{\pi^2} \zeta(3) g T^3 & \text{bosons}
    \end{cases} \\
    \rho &= \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi p^3 \, dp}{e^p/T \pm 1} = \begin{cases} 
    \frac{7}{8} \pi^2 \frac{g T^4}{30} & \text{fermions} \\
    \frac{\pi^2}{2} \frac{g T^4}{30} & \text{bosons}
    \end{cases} \\
    p &= \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi p^3 \, dp}{e^p/T \pm 1} = \frac{1}{3} \rho \approx \begin{cases} 
    1.0505 n T & \text{fermions} \\
    0.9004 n T & \text{bosons}
    \end{cases}
\end{align*}

(12)

(13)

(14)

For the average particle energy we get

\begin{align*}
    \langle E \rangle &= \frac{\rho}{n} = \begin{cases} 
    \frac{7\pi^4}{180 \zeta(3)} T & \approx 3.151 T \quad \text{fermions} \\
    \frac{\pi^4}{30 \zeta(3)} T & \approx 2.701 T \quad \text{bosons}
    \end{cases}
\end{align*}

(15)

In the above, $\zeta$ is the Riemann zeta function, and $\zeta(3) \equiv \sum_{n=1}^\infty (1/n^3) = 1.20206$.

If the chemical potential $\mu = 0$, there are equal numbers of particles and antiparticles. If $\mu \neq 0$, we find for fermions in the ultrarelativistic limit $T \gg m$ (i.e., for $m = 0$, but $\mu \neq 0$) the “net particle number”

\begin{align*}
    n - \bar{n} &= \frac{g}{(2\pi)^3} \int_0^\infty dp \, 4\pi p^2 \left( \frac{1}{e^{(p-\mu)/T} + 1} - \frac{1}{e^{(p+\mu)/T} + 1} \right) \\
    &= \frac{g T^3}{6\pi^2} \left( \pi^2 \left( \frac{\mu}{T} \right)^2 + \left( \frac{\mu}{T} \right)^3 \right)
\end{align*}

(16)

and the total energy density

\begin{align*}
    \rho + \bar{\rho} &= \frac{g}{(2\pi)^3} \int_0^\infty dp \, 4\pi p^3 \left( \frac{1}{e^{(p-\mu)/T} + 1} + \frac{1}{e^{(p+\mu)/T} + 1} \right) \\
    &= \frac{7}{8} \frac{\pi^2}{15} T^4 \left( 1 + \frac{30}{7\pi^2} \left( \frac{\mu}{T} \right)^2 + \frac{15}{7\pi} \left( \frac{\mu}{T} \right)^4 \right)
\end{align*}

(17)

Note that the last forms in Eqs. (16) and (17) are exact, not just truncated series. (The difference $n - \bar{n}$ and the sum $\rho + \bar{\rho}$ lead to a nice cancellation between the two integrals. We don’t get such an elementary form for the individual $n$, $\bar{n}$, $\rho$, $\bar{\rho}$, or the sum $n + \bar{n}$ and the difference $\rho - \bar{\rho}$ when $\mu \neq 0$.)

In the nonrelativistic limit, $T \ll m$ and $T \ll m - \mu$, the typical kinetic energies are much below the mass $m$, so that we can approximate $E = m + p^2/2m$. The second condition, $T \ll m - \mu$, leads to occupation numbers $\ll 1$, a dilute system. This second condition is usually satisfied in cosmology when the first one is. (It is violated in systems of high density, like white dwarf stars and neutron stars.) We can then approximate

\[ e^{(E-\mu)/T} \pm 1 \approx e^{(E-\mu)/T} \]

(18)
so that the boson and fermion expressions become equal, and we get (exercise)

\[ n = g \left( \frac{mT}{2\pi} \right)^{3/2} e^{-\frac{m-\mu}{T}} \quad (19) \]

\[ \rho = n \left( m + \frac{3T}{2} \right) \quad (20) \]

\[ p = nT \ll \rho \quad (21) \]

\[ \langle E \rangle = m + \frac{3T}{2} \quad (22) \]

\[ n - \bar{n} = 2g \left( \frac{mT}{2\pi} \right)^{3/2} e^{-\frac{m}{T}} \sinh \frac{\mu}{T} \quad (23) \]

In the general case, where neither \( T \ll m \), nor \( T \gg m \), the integrals don’t give elementary functions, but \( n(T), \rho(T), \) etc. need to be calculated numerically for the region \( T \sim m \).

By comparing the ultrarelativistic \( (T \gg m) \) and nonrelativistic \( (T \ll m) \) limits we see that the number density, energy density, and pressure of a particle species falls exponentially as the temperature falls below the mass of the particle. What happens is that the particles and antiparticles annihilate each other. (Other reactions may also be involved, and if these particles are unstable, also their decay contributes to their disappearance.) At higher temperatures these annihilation reactions are also constantly taking place, but they are balanced by particle-antiparticle pair production. At lower temperatures the thermal particle energies are no more sufficient for pair production. This particle-antiparticle annihilation takes place mainly (about 80\%) during the temperature interval \( T = m \rightarrow \frac{1}{6} m \). See Fig. 1. It is thus not an instantaneous event, but takes several Hubble times.

![Figure 1: The fall of energy density of a particle species, with mass \( m \), as a function of temperature (decreasing to the right).](image)

### 4.2 Primordial soup

We shall now apply the thermodynamics discussed in the previous section to the evolution of the early universe.

The primordial soup initially consists of all the different species of elementary particles. Their masses range from the heaviest known elementary particle, the top quark \((m = 173 \text{ GeV})\) down to the lightest particles, the electron \((m = 511 \text{ keV})\), the neutrinos \((m = ?)\) and the...
photon \((m = 0)\). In addition to the particles of the standard model of particle physics (given in Table 1), there are other, so far undiscovered, species of particles, at least those that make up the CDM. As the temperature falls, the various particle species become nonrelativistic and annihilate at different times.

Another central theme is decoupling: as the number densities and particle energies fall with the expansion, some reaction rates become too low to keep up with the changing equilibrium and therefore some quantities are “frozen” at their pre-decoupling values. We will encounter neutrino and photon decoupling later in this chapter; decoupling is also important in BBN (Chapter 5) and for dark matter (Chapter 6).

Table 1: The particles in the standard model of particle physics

*Particle Data Group, 2018*

<table>
<thead>
<tr>
<th>Particle Type</th>
<th>Mass/Energy</th>
<th>Spin</th>
<th>Internal Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quarks</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(t)</td>
<td>173.0 ± 0.4 GeV</td>
<td>(\frac{1}{2})</td>
<td>6</td>
</tr>
<tr>
<td>(b)</td>
<td>4.15–4.22 GeV</td>
<td>(\frac{1}{2})</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>1.27 ± 0.03 GeV</td>
<td>(\frac{1}{2})</td>
<td></td>
</tr>
<tr>
<td>(s)</td>
<td>92–104 MeV</td>
<td>(\frac{1}{2})</td>
<td></td>
</tr>
<tr>
<td>(d)</td>
<td>4.4–5.2 MeV</td>
<td>(\frac{1}{2})</td>
<td></td>
</tr>
<tr>
<td>(u)</td>
<td>1.8–2.7 MeV</td>
<td>(\frac{1}{2})</td>
<td></td>
</tr>
<tr>
<td><strong>Gluons</strong></td>
<td>8 massless bosons</td>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td><strong>Leptons</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\tau^-)</td>
<td>1776.86 ± 0.12 MeV</td>
<td>(\frac{1}{2})</td>
<td>2</td>
</tr>
<tr>
<td>(\mu^-)</td>
<td>105.658 MeV</td>
<td>(\frac{1}{2})</td>
<td>2</td>
</tr>
<tr>
<td>(e^-)</td>
<td>510.999 keV</td>
<td>(\frac{1}{2})</td>
<td>2</td>
</tr>
<tr>
<td>(\nu_\tau)</td>
<td>&lt; 2 eV</td>
<td>(\frac{1}{2})</td>
<td>1</td>
</tr>
<tr>
<td>(\nu_\mu)</td>
<td>&lt; 2 eV</td>
<td>(\frac{1}{2})</td>
<td>1</td>
</tr>
<tr>
<td>(\nu_e)</td>
<td>&lt; 2 eV</td>
<td>(\frac{1}{2})</td>
<td>1</td>
</tr>
<tr>
<td><strong>Electroweak gauge bosons</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(W^+)</td>
<td>80.379 ± 0.012 GeV</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>(Z^0)</td>
<td>91.1876±0.0021 GeV</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>0 (&lt; 1 \times 10^{-18} eV)</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td><strong>Higgs boson (SM)</strong></td>
<td>(H^0)</td>
<td>125.18 ± 0.16 GeV</td>
<td>0</td>
</tr>
</tbody>
</table>

The mass limits for neutrinos come from a direct laboratory upper limit for \(\nu_e\) and evidence from neutrino oscillations that the differences in neutrino masses are much smaller. We can use cosmology to put tighter limits to neutrino masses. Neutrinos are special in that the antineutrino is just the other spin state of the neutrino. Therefore we put \(g = 1\) for their internal degrees of freedom when we count antineutrinos separately.
According to the Friedmann equation the expansion of the universe is governed by the total energy density

\[ \rho(T) = \sum_i \rho_i(T), \]

where \( i \) runs over the different particle species. Since the energy density of relativistic species is much greater than that of nonrelativistic species, it suffices to include the relativistic species only. (This is true in the early universe, during the radiation-dominated era, but not at later times. Eventually the rest masses of the particles left over from annihilation begin to dominate and we enter the matter-dominated era.) Thus we have

\[ \rho(T) = \frac{\pi^2}{30} g_*(T) T^4, \]  

(24)

where

\[ g_*(T) = g_b(T) + \frac{7}{8} g_f(T), \]

and \( g_b = \sum_i g_i \) over relativistic bosons and \( g_f = \sum_i g_i \) over relativistic fermions. These results assume thermal equilibrium. For pressure we have \( p(T) \approx \frac{1}{3} \rho(T) \).

The above is a simplification of the true situation: Since the annihilation takes a long time, often the annihilation of some particle species is going on, and the contribution of this species disappears gradually. Using the exact formula for \( \rho \) we define the effective number of degrees of freedom \( g_*(T) \) by

\[ g_*(T) \equiv \frac{30}{\pi^2} \frac{\rho}{T^4}. \]  

(25)

We can also define

\[ g_{*p}(T) \equiv \frac{90}{\pi^2} \frac{p}{T^4} \approx g_*(T). \]  

(26)

These can then be calculated numerically (see Figure 1).

We see that when there are no annihilations taking place, \( g_{*p} = g_* = \text{const} \Rightarrow p = \frac{1}{3} \rho \Rightarrow \rho \propto a^{-4} \) and \( \rho \propto T^4 \), so that \( T \propto a^{-1} \). Later in this chapter we shall calculate the \( T(a) \) relation more exactly (including the effects of annihilations).

For \( T > m_t = 173 \text{ GeV} \), all known particles are relativistic. Adding up their internal degrees of freedom we get

\[ g_b = 28 \quad \text{gluons } 8 \times 2, \text{ photons } 2, \text{ } W^\pm \text{ and } Z^0 \text{ } 3 \times 3, \text{ and Higgs } 1 \]
\[ g_f = 90 \quad \text{quarks } 12 \times 6, \text{ charged leptons } 6 \times 2, \text{ neutrinos } 3 \times 2 \]
\[ g_* = 106.75. \]

The electroweak (EW) transition\(^3\) took place close to this time \((T_c \sim 100 \text{ GeV})\). It appears that \( g_* \) was the same before and after this transition. Going to earlier times and higher temperatures, we expect \( g_* \) to get larger than 106.75 as new physics (new unknown particle species) comes to play.\(^4\)

\( ^3 \)This is usually called the electroweak phase transition, but the exact nature of the transition is not known. Technically it may be a cross-over rather than a phase transition, meaning that it occurs over a temperature range rather than at a certain critical temperature \( T_c \).

\( ^4 \)A popular form of such new physics is supersymmetry, which provides supersymmetric partners, whose spin differ by \( \frac{1}{2} \), for the known particle species, so that fermions have supersymmetric boson partners and bosons have supersymmetric fermion partners. Since these partners have not been so far observed, supersymmetry must be broken, allowing these partners to have much higher masses. In the minimal supersymmetric standard model (MSSM) the new internal degrees of freedom are as follows: Spin-0 bosons (scalars): sleptons \( 9 \cdot 2 = 18 \), squarks \( 6 \cdot 2 \cdot 2 \cdot 3 = 72 \) (although there is only one spin degree instead of 2, there is another degree of freedom, so that we get the same \( 18+72 \) as for leptons and quarks), and a new complex Higgs doublet \( 2 \cdot 2 = 4 \). Spin-\( \frac{1}{2} \) fermions: neutralinos \( 4 \cdot 2 = 8 \), charginos \( 2 \cdot 2 \cdot 2 \) (two charge degrees and two spin degrees), and gluinos \( 8 \cdot 2 = 16 \). This gives \( g_* = 106.75 + 94 + \frac{5}{2} \cdot 32 = 228.75 \). Other supersymmetric models have somewhat more degrees of freedom but some of the new degrees of freedom may be very heavy \((10^{10} \ldots 10^{16} \text{ GeV})\).\(^{[1]}\)
Let us now follow the history of the universe starting at the time when the EW transition has already happened. We have $T \sim 100 \text{ GeV}$, $t \sim 20 \text{ ps}$, and the $t$ quark annihilation is on the way. The Higgs boson and the gauge bosons $W^\pm$, $Z^0$ annihilate next. At $T \sim 10 \text{ GeV}$, we have $g_* = 86.25$. Next the $b$ and $c$ quarks annihilate and then the $\tau$ meson, so that $g_* = 61.75$.

### 4.3 QCD transition

Before $s$ quark annihilation would take place, something else happens: the QCD transition (also called the quark–hadron transition). This takes place at $T \sim 150 \text{ MeV}$, $t \sim 20 \mu\text{s}$. The temperature and thus the quark energies have fallen so that the quarks lose their so called asymptotic freedom, which they have at high energies. The interactions between quarks and gluons (the strong nuclear force, or the color force) become important (so that the formulae for the energy density in Sec. 4.1 no longer apply) and soon a phase transition takes place. There are no more free quarks and gluons; the quark-gluon plasma has become a hadron gas. The quarks and gluons have formed bound three-quark systems, called baryons, and quark-antiquark pairs, called mesons. The lightest baryons are the nucleons: the proton and the neutron. The lightest mesons are the pions: $\pi^\pm$, $\pi^0$. Baryons are fermions, mesons are bosons.

There are very many different species of baryons and mesons, but all except pions are non-relativistic below the QCD transition temperature. Thus the only particle species left in large numbers are the pions, muons, electrons, neutrinos, and the photons. For pions, $g = 3$, so now $g_* = 17.25$. 

Figure 2: The functions $g_*(T)$ (solid), $g_{sp}(T)$ (dashed), and $g_{*s}(T)$ (dotted) calculated for the standard model particle content.
Table 2: History of $g_*(T)$

<table>
<thead>
<tr>
<th>Temperature (MeV)</th>
<th>Particle Annihilation</th>
<th>$g_*$ after Annihilation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T \sim 200$</td>
<td>all present</td>
<td>106.75</td>
</tr>
<tr>
<td>$T \sim 100$</td>
<td>EW transition (no effect)</td>
<td></td>
</tr>
<tr>
<td>$T &lt; 170$</td>
<td>top annihilation</td>
<td>96.25</td>
</tr>
<tr>
<td>$T &lt; 80$</td>
<td>$W^\pm, Z^0, H^0$</td>
<td>86.25</td>
</tr>
<tr>
<td>$T &lt; 4$</td>
<td>bottom</td>
<td>75.75</td>
</tr>
<tr>
<td>$T &lt; 1$</td>
<td>charm, $\tau^-$</td>
<td>61.75</td>
</tr>
<tr>
<td>$T \sim 150$</td>
<td>QCD transition</td>
<td>17.25</td>
</tr>
<tr>
<td>$T &lt; 100$</td>
<td>$\pi^\pm, \pi^0, \mu^-$</td>
<td>10.75</td>
</tr>
<tr>
<td>$T &lt; 4$</td>
<td>bottom</td>
<td></td>
</tr>
<tr>
<td>$T &lt; 1$</td>
<td>charm, $\tau^-$</td>
<td></td>
</tr>
<tr>
<td>$T \sim 100$</td>
<td>$\pi^\pm, \pi^0, \mu^-$</td>
<td>47.5</td>
</tr>
<tr>
<td>$T &lt; 500$</td>
<td>$e^-$ annihilation</td>
<td>2 + 5.25(4/11)^{4/3} = 3.36</td>
</tr>
</tbody>
</table>

This table gives what value $g_*(T)$ would have after the annihilation of a particle species is over assuming the annihilation of the next species had not begun yet. In reality they overlap in many cases. The temperature value at the left is the approximate mass of the particle in question and indicates roughly when annihilation begins. The temperature is much smaller when the annihilation is over. Therefore top annihilation is placed after the EW transition. The top quark receives its mass in the EW transition, so annihilation only begins after the transition.

4.4 Neutrino decoupling and electron-positron annihilation

Soon after the QCD phase transition the pions and muons annihilate and for $T = 20$ MeV $\rightarrow 1$ MeV, $g_* = 10.75$. Next the electrons annihilate, but to discuss the $e^+e^-$-annihilation we need more physics.

So far we have assumed that all particle species have the same temperature, i.e., the interactions among the particles are able to keep them in thermal equilibrium. Neutrinos, however, feel the weak interaction only. The weak interaction is actually not so weak when particle energies are close to the masses of the $W^\pm$ and $Z^0$ bosons, which mediate the weak interaction. But as the temperature falls, the weak interaction becomes rapidly weaker and weaker. Finally, close to $T \sim 1$ MeV, the neutrinos decouple, after which they move practically freely without interactions.

The momentum of a freely moving neutrino redshifts as the universe expands,

$$p(t_2) = (a_1/a_2)p(t_1).$$

(27)

From this follows that neutrinos stay in kinetic equilibrium. This is true in general for ultrarelativistic ($m \ll T \Rightarrow p = E$) noninteracting particles. Let us show this:

At time $t_1$ a phase space element $d^3p_1dV_1$ contains

$$dN = \frac{g}{(2\pi)^3} f(\vec{p}_1)d^3p_1dV_1$$

(28)

particles, where

$$f(\vec{p}_1) = \frac{1}{e(\vec{p}_1-\vec{\mu}_1)/T_1 \pm 1}$$

is the distribution function at time $t_1$. At time $t_2$ these same $dN$ particles are in a phase space element $d^3p_2dV_2$. Now how is the distribution function at $t_2$, given by

$$\frac{g}{(2\pi)^3} f(\vec{p}_2) = \frac{dN}{d^3p_2dV_2},$$

(28)
Figure 3: The expansion of the universe increases the volume element $dV$ and decreases the momentum space element $d^3p$ so that the phase space element $d^3pdV$ stays constant.

related to $f(\tilde{p})$? Since $d^3p_2 = (a_1/a_2)^3d^3p_1$ and $dV_2 = (a_2/a_1)^3dV_1$, we have

$$dN = \frac{g}{(2\pi)^3} \frac{d^3p_1}{e^{(p_1 - \mu_1)/T_1} \pm 1} dV_1$$

(dN evaluated at $t_1$)

$$= \frac{g}{(2\pi)^3} \frac{(a_2/a_1)^3 d^3p_2 (a_1/a_2)^3 dV_2}{e^{(p_2 - \mu_1)/T_1} \pm 1}$$

(rewritten in terms of $p_2$, $dp_2$, and $dV_2$) (29)

$$= \frac{g}{(2\pi)^3} \frac{d^3p_2 dV_2}{e^{(p_2 - \mu_2)/T_2} \pm 1}$$

(rewritten in terms of $p_2$, $dp_2$, and $dV_2$)

$$= \frac{g}{(2\pi)^3} \frac{d^3p_2 dV_2}{(a_2/a_1)^3 e^{(p_2 - \mu_2)/T_2} \pm 1}$$

(rewritten in terms of $p_2$, $dp_2$, and $dV_2$) (30)

$$= \frac{g}{(2\pi)^3} \frac{d^3p_2 dV_2}{(a_2/a_1)^3 e^{(p_2 - \mu_2)/T_2} \pm 1}$$

(rewritten in terms of $p_2$, $dp_2$, and $dV_2$)

$$= \frac{g}{(2\pi)^3} \frac{d^3p_2 dV_2}{(a_2/a_1)^3 e^{(p_2 - \mu_2)/T_2} \pm 1}$$

(rewritten in terms of $p_2$, $dp_2$, and $dV_2$)

where $\mu_2 \equiv (a_1/a_2)\mu_1$ and $T_2 \equiv (a_1/a_2)T_1$. Thus the particles keep the shape of a thermal distribution; the temperature and the chemical potential just redshift $\propto a^{-1}$. (Exercise: For nonrelativistic particles, $m \gg T \Rightarrow E = m + p^2/2m$, there is a corresponding, but different result. Derive this.)

Thus for as long as $T \propto a^{-1}$ for the particle soup, the neutrino distribution evolves exactly as if it were in thermal equilibrium with the soup, i.e., $T_\nu = T$. However, annihilations will cause a deviation from $T \propto a^{-1}$. The next annihilation event is the electron-positron annihilation.

The easiest way to obtain the relation between the temperature $T$ and the scale factor $a$ is to use entropy conservation.

From the fundamental equation of thermodynamics,

$$E = TS - pV + \sum \mu_i N_i$$

we have

$$s = \frac{\rho + p - \sum \mu_i n_i}{T},$$

for the entropy density $s \equiv S/V$. Since $|\mu_i| \ll T$, and the relativistic species dominate, we approximate

$$s = \frac{\rho + p}{T} = \begin{cases} \frac{7\pi^2}{180} g T^3 & \text{fermions} \\ \frac{2\pi^2}{45} g T^3 & \text{bosons} \end{cases}$$

(31)
Adding up all the relativistic species and allowing now for the possibility that some species may have a kinetic temperature $T_i$, which differs from the temperature $T$ of those species which remain in thermal equilibrium, we get

$$\rho(T) = \frac{\pi^2}{30} g_*(T) T^4$$
$$s(T) = \frac{2\pi^2}{45} g_{ss}(T) T^3,$$

(32)

where now

$$g_*(T) = \sum_{\text{bos}} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{\text{fer}} g_i \left( \frac{T_i}{T} \right)^4$$

$$g_{ss}(T) = \sum_{\text{bos}} g_i \left( \frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{\text{fer}} g_i \left( \frac{T_i}{T} \right)^3,$$

(33)

and the sums are over all relativistic species of bosons and fermions.

If some species are “semirelativistic”, i.e., $m = \mathcal{O}(T)$, $\rho(T)$ and $s(T)$ are to be calculated from the integral formulae in Sec. 4.1, and Eq. (32) defines $g_*(T)$ and $g_{ss}(T)$.

For as long as all species have the same temperature and $p \approx \frac{1}{3} \rho$, we have

$$g_{ss}(T) \approx g_*(T).$$

(34)

The electron annihilation, however, forces us to make a distinction between $g_*(T)$ and $g_{ss}(T)$.

According to the second law of thermodynamics the total entropy of the universe never decreases; it either stays constant or increases. An increase in entropy is always related to a deviation from thermodynamic equilibrium. It turns out that any entropy production in the various known processes in the universe is totally insignificant compared to the total entropy of the universe\(^5\), which is huge, and dominated by the relativistic species. Thus it is an excellent approximation to treat the expansion of the universe as \textit{adiabatic}, so that the total entropy stays constant, i.e.,

$$d(sa^3) = 0.$$

(35)

This now gives us the relation between $a$ and $T$,

$$g_{ss}(T) T^3 a^3 = \text{const}.$$  

(36)

We shall have much use for this formula.

In the electron annihilation $g_{ss}$ changes from

$$g_{ss} = g_* = 2 + 3.5 + 5.25 = 10.75$$

(37)

$$\gamma \quad e^\pm \quad \nu$$

to

$$g_{ss} = 2 + 5.25 \left( \frac{T_\nu}{T} \right)^3,$$

(38)

where

$$T_\nu^3 a^3 = \text{const} = T^3 a^3(\text{before annihilation}).$$

(39)

---

\(^5\)There may be exceptions to this in the very early universe, most notably \textit{inflation}, where essentially all the entropy of the universe supposedly was produced.
(since the neutrinos have decoupled, \( T_{\nu} \) redshifts \( T_{\nu} \propto a^{-1} \)). As the number of relativistic degrees of freedom is reduced, energy density and entropy are transferred from electrons and positrons to photons, but not to neutrinos, in the annihilation reactions

\[
e^+ + e^- \rightarrow \gamma + \gamma.
\]

The photons are thus heated (the photon temperature does not fall as much) relative to neutrinos.

Dividing Eq. (36) with Eq. (39) we get that

\[
g_{\ast s}(T) \left( \frac{T}{T_{\nu}} \right)^3 = \text{const}
\]

or (Eqs. (37) and (38))

\[
10.75 = 2 \left( \frac{T}{T_{\nu}} \right)^3 + 5.25 \quad \text{(before = after)}
\]

from which we solve the neutrino temperature after \( e^+e^-\)-annihilation,\(^6\)

\[
T_{\nu} = \left( \frac{4}{11} \right)^{1/3} T = 0.714 T
\]

\[
g_{\ast s}(T) = 2 + 5.25 \cdot \frac{4}{11} = 3.909 \quad \text{(40)}
\]

\[
g_\ast(T) = 2 + 5.25 \left( \frac{4}{11} \right)^{3/4} = 3.363.
\]

These relations remain true for the photon+neutrino background as long as the neutrinos stay ultrarelativistic \((m_\nu \ll T)\). It used to be the standard assumption that neutrinos are massless or that their masses are so small that they can be ignored, in which case the above relation would apply even today, when the photon (the CMB) temperature is \( T = T_0 = 2.725 \) K = 0.2348 meV, giving the neutrino background the temperature \( T_{\nu0} = 0.714 \cdot 2.725 \) K = 1.945 K = 0.1676 meV today. However, \textit{neutrino oscillation} experiments suggest a neutrino mass in the meV range, so that the neutrino background could be nonrelativistic today. In any case, the CMB (photon) temperature keeps redshifting as \( T \propto a^{-1} \), so we can use Eq. (36) to relate the scale factor \( a \) and the CMB temperature \( T \), keeping \( g_{\ast s}(T) = 3.909 \) all the way to the present time (and into the future).

Regardless of the question of neutrino masses, these relativistic backgrounds do not dominate the energy density of the universe any more today (photons + neutrinos still dominate the entropy density), as we shall discuss in Sec. 4.6.

### 4.5 Time scale of the early universe

The curvature term \( K/a^2 \) and dark energy can be ignored in the early universe, so the metric is

\[
ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \right].
\]

\(^6\)To be more precise, neutrino decoupling was not complete when \( e^+e^-\)-annihilation began; so that some of the energy and entropy leaked to the neutrinos. Therefore the neutrino energy density after \( e^+e^-\)-annihilation is about 1.3% higher (at a given \( T \)) than the above calculation gives. The neutrino distribution also deviates slightly from kinetic equilibrium.
Figure 4: The evolution of the energy density, or rather, \( g_*(T) \), and its different components through electron-positron annihilation. Since \( g_*(T) \) is defined as \( \rho/(\pi^2 T^4/30) \), where \( T \) is the photon temperature, the photon contribution appears constant. If we had plotted \( \rho/(\pi^2 T^4/30) \propto \rho_a \), instead, the neutrino contribution would appear constant, and the photon contribution would increase at the cost of the electron-positron contribution, which would better reflect what is going on.

and the Friedmann equation is

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho(T) = \frac{8\pi G}{3} \frac{\pi^2}{30} g_*(T) T^4. \tag{42}
\]

To integrate this equation exactly we would need to calculate numerically the function \( g_*(T) \) with all the annihilations\(^7\). For most of the time, however, \( g_*(T) \) is changing slowly, so we can approximate \( g_*(T) = \text{const} \). Then \( T \propto a^{-1} \) and \( H(t) = H(t_1)(a_1/a)^2 \). Thus

\[
dt = \frac{da}{a} H^{-1}(t_1) \left( \frac{a}{a_1} \right)^2 \quad \Rightarrow \quad t_1 = H^{-1}(t_1) \int_0^{a_1} \left( \frac{a}{a_1} \right) \frac{da}{a_1} = \frac{1}{2} H^{-1}(t_1)
\]

and we get the relation

\[
t = \frac{1}{2} H^{-1} = \sqrt{\frac{45}{16\pi^3 G \sqrt{g_*}}} \frac{T^{-2}}{g_*} = 0.301 g_*^{-1/2} \frac{m_{\text{Pl}}}{T^2} = \frac{2.4}{\sqrt{g_*}} \left( \frac{T}{\text{MeV}} \right)^{-2} \quad \text{s}
\tag{43}
\]

between the age of the universe \( t \) and the Hubble parameter \( H \). Here

\[
m_{\text{Pl}} = \frac{1}{\sqrt{G}} = 1.22 \times 10^{19} \text{ GeV}
\]

is the Planck mass. Thus

\[
a \propto T^{-1} \propto t^{1/2}.
\]

\(^7\)During electron annihilation one needs to calculate \( g_{\text{es}}(T) \) also, to get \( T_\nu(T) \), needed for \( g_*(T) \).
Except for a few special stages (like the QCD transition) the error from ignoring the time-
dependence of $g_\ast(T)$ is small, since the time scales of earlier events are so much shorter, so the
approximate result, Eq. (43), will be sufficient for us, as far as the time scale is concerned, when
we use for each time $t$ the value of $g_\ast$ at that time. But for the relation between $a$ and $T$, we
need to use the more exact result, Eq. (36). Table 4 gives the times of the different events in
the early universe.

Let us calculate (was already done in Chapter 3) the distance to the horizon $d_{\text{hor}}(t_1) = a_1 r_{\text{hor}}(t_1)$ at a given time $t_1$. For a radial light ray $dt = a(t)dr$ and from above $a(t) = (t/t_1)^{1/2}a_1$. Thus
\[
\frac{t_1^{1/2}}{t_0^{1/2}} \int_{t_0}^{t_1} \frac{dt}{t^{1/2}} = a_1 \int_0^{r_{\text{hor}}} dr \quad \Rightarrow \quad 2t_1 = a_1 r_{\text{hor}} = d_{\text{hor}}(t_1)
\]
and we find for the horizon
\[
d_{\text{hor}} = 2t = H \left( t_0 \right)^{-1}.
\]
Thus in the radiation-dominated early universe the distance to the horizon is equal to the Hubble
length.

### 4.6 Matter

We noted that the early universe is dominated by the relativistic particles, and we can forget
the nonrelativistic particles when we are considering the dynamics of the universe. We followed
one species after another becoming nonrelativistic and disappearing from the picture, until only
photons (the cosmic background radiation) and neutrinos were left, and even the latter of these
had stopped interacting.

We must now return to the question what happened to the nucleons and the electrons.
We found that they annihilated with their antiparticles when the temperature fell below their
respective rest masses. For nucleons, the annihilation began immediately after they were formed
in the QCD phase transition. There were however slightly more particles than antiparticles, and
this small excess of particles was left over. (This must be so since we observe electrons and
nucleons today). This means that the chemical potential $\mu_B$ associated with baryon number
differs from zero (is positive). Baryon number is a conserved quantity. Since nucleons are the
lightest baryons, the baryon number resides today in nucleons (protons and neutrons; since
the proton is lighter than the neutron, free neutrons have decayed into protons, but there are
neutrons in atomic nuclei, whose mass/baryon is even smaller). The universe is electrically
neutral, and the negative charge lies in the electrons, the lightest particles with negative charge.
Therefore the number of electrons must equal the number of protons.

The number densities etc. of the electrons and the nucleons we get from the equations of
Sec. 4.1. But what is the chemical potential $\mu$ in them? For each species, we get $\mu(T)$ from the
conserved quantities.\(^8\) The baryon number resides in the nucleons,
\[
n_B = n_N - n_\bar{N} = n_p + n_n - n_\bar{p} - n_\bar{n}.
\]
Let us define the parameter $\eta$, the baryon-photon ratio today,
\[
\eta \equiv \frac{n_B(t_0)}{n_\gamma(t_0)}.
\]
\(^8\)In general, the recipe to find how the thermodynamical parameters, temperature and the chemical poten-
tials, evolve in the expanding FRW universe, is to use the conservation laws of the conserved numbers, entropy
conservation, and energy continuity, to find how the number densities and energy densities must evolve. The
thermodynamical parameters will then evolve to satisfy these requirements.
From observations we know that \( \eta = 10^{-10} - 10^{-9} \). Since baryon number is conserved, \( n_B V \propto n_B a^3 \) stays constant, so
\[
n_B \propto a^{-3}. \quad (47)
\]
After electron annihilation \( n_\gamma \propto a^{-3} \), so we get
\[
n_B(T) = \eta m_\gamma = \eta \frac{2\zeta(3)}{\pi^2} T^3 \quad \text{for } T \ll m_e, \quad (48)
\]
and for all times (as long as the universe expands adiabatically and the baryon number is conserved), using Eqs. (36), (47), and (48),
\[
n_B(T) = \eta \frac{2\zeta(3)}{\pi^2} \frac{g_{*s}(T)}{g_{*s}(T_0)} T^3. \quad (49)
\]
For \( T < 10 \text{ MeV} \) we have in practice
\[
n_N \ll n_N \quad \text{and} \quad n_N \equiv n_n + n_p = n_B.
\]

We shall later (Chapter 5) discuss big bang nucleosynthesis—how the protons and neutrons formed atomic nuclei. Approximately one quarter of all nucleons (all neutrons and roughly the same number of protons) form nuclei \((A > 1)\) and three quarters remain as free protons. Let us denote by \( n_p^* \) and \( n_n^* \) the number densities of protons and neutrons including those in nuclei (and also those in atoms), whereas we shall use \( n_p \) and \( n_n \) for the number densities of free protons and neutrons. Thus we write instead
\[
n_N^* \equiv n_n^* + n_p^* = n_B.
\]

In the same manner, for \( T < 10 \text{ keV} \) we have
\[
n_e^+ \ll n_e^- \quad \text{and} \quad n_e^- = n_p^*.
\]
At this time \((T \sim 10 \text{ keV} \to 1 \text{ eV})\) the universe contains a relativistic photon and neutrino background ("radiation") and nonrelativistic free electrons, protons, and nuclei ("matter"). Since \( \rho \propto a^{-4} \) for radiation, but \( \rho \propto a^{-3} \) for matter, the energy density in radiation falls eventually below the energy density in matter—the universe becomes matter-dominated.

The above discussion is in terms of the known particle species. Today there is much indirect observational evidence for the existence of what is called cold dark matter (CDM), which is supposedly made out of some yet undiscovered species of particles (this is discussed in Chapter 6). The CDM particles should be very weakly interacting (they decouple early), and their energy density contribution should be small when we are well in the radiation-dominated era, so they do not affect the above discussion much. They become nonrelativistic early and they are supposed to dominate the matter density of the universe (there appears to be about five times as much mass in CDM as in baryons). Thus the CDM causes the universe to become matter-dominated earlier than if the matter consisted of nucleons and electrons only. The CDM will be important later when we discuss (in Cosmology II) the formation of structure in the universe. The time of matter-radiation equality \( t_{eq} \) is calculated in an exercise at the end of this chapter.

### 4.7 Neutrino masses

The observed phenomenon of neutrino oscillations, where neutrinos change their flavor (i.e., whether they are \( \nu_e, \nu_\mu, \text{ or } \nu_\tau \) periodically, is an indication of differences in the neutrino masses and therefore the neutrinos cannot all be massless. The oscillation phenomenon is a quantum mechanical effect, and is due to the mass eigenstates of neutrinos (a quantum state with definite
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\[ \sum m_i = m_1 + m_2 + m_3 \]

<table>
<thead>
<tr>
<th>Normal</th>
<th>Inverted</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 )</td>
<td>( m_3 )</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>( m_1 )</td>
</tr>
<tr>
<td>( m_3 )</td>
<td>( m_2 )</td>
</tr>
<tr>
<td>( \sum m_i )</td>
<td>( \sum m_i )</td>
</tr>
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<td>0</td>
</tr>
<tr>
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<td>101 meV</td>
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<tr>
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<td>111.8 meV</td>
</tr>
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</tr>
<tr>
<td>6 eV</td>
<td>6 eV</td>
</tr>
</tbody>
</table>

Table 3: Possibilities for neutrino masses.

Two of the mass eigenstates, labeled \( m_1 \) and \( m_2 \), are thus close to each other and \( m_1 < m_2 \); but we do not know whether the third mass eigenstate has a larger or smaller mass. These two possibilities are called the normal (\( m_1 < m_2 < m_3 \)) and inverted (\( m_3 < m_1 < m_2 \)) hierarchies. The neutrino mixing matrix, which relates the mass and flavor eigenstates, is not known well, but it appears that \( m_2 \) is a roughly equal mixture of all three flavors, and if we have the normal hierarchy, \( m_3 \) is mostly \( \nu_\mu \) and \( \nu_\tau \).[2]

Since we have a laboratory upper limit \( m < 2 \) eV for \( \nu_e \), the smallest of these mass eigenstates must be \( < 2 \) eV. (Measurement of the mass of a neutrino flavor projects the flavor state into a mass state, giving \( m_1, m_2, \) or \( m_3 \) with different probabilities; the upper limit presumably refers to the mass expectation value of the flavor state.) To have an idea what these \( \Delta m^2 \) mean for neutrino masses, consider three possibilities for the lowest mass eigenstate: \( m = 0 \), \( m = 100 \) meV, and \( m = 2 \) eV. This gives Table 3 and we conclude that the sum of the three neutrino masses must lie between \( \sim 0.06 \) eV and \( \sim 6 \) eV, and that if we have the inverted hierarchy, it should be at least 0.1 eV. The smallest possibility, where \( m_1 \ll m_2 \) and \( \sum m_i = 0.06 \) eV, is perhaps the most natural one and is considered as part of the standard model of cosmology (and the other possibilities are “extensions” of this standard model).

4.8 Recombination

Radiation (photons) and matter (electrons, protons, and nuclei) remained in thermal equilibrium for as long as there were lots of free electrons. When the temperature became low enough the electrons and nuclei combined to form neutral atoms (recombination), and the density of free electrons fell sharply. The photon mean free path grew rapidly and became longer than the horizon distance. Thus the universe became transparent. Photons and matter decoupled, i.e., their interaction was no more able to maintain them in thermal equilibrium with each other. After this, by \( T \) we refer to the photon temperature. Today, these photons are the CMB, and \( T = T_0 = 2.725 \) K. (After photon decoupling, the matter temperature fell at first faster than the photon temperature, but structure formation then heated up the matter to different temperatures at different places.)

To simplify the discussion of recombination, let us forget other nuclei than protons (in reality over 90% (by number) of the nuclei are protons, and almost all the rest are \(^4\)He nuclei). Let us denote the number density of free protons by \( n_p \), free electrons by \( n_e \), and hydrogen atoms
by $n_H$. Since the universe is electrically neutral, $n_p = n_e$. The conservation of baryon number gives $n_B = n_p + n_H$. From Sec. 4.1 we have

$$n_i = g_i \left( \frac{m_i T}{2\pi} \right)^{3/2} e^{-\frac{\mu_i - m_i}{T}}. \quad (51)$$

For as long as the reaction

$$p + e^- \leftrightarrow H + \gamma \quad (52)$$

is in chemical equilibrium the chemical potentials are related by $\mu_p + \mu_e = \mu_H$ (since $\mu_\gamma = 0$). Using this we get the relation

$$n_H = \frac{g_H}{g_p g_e} n_p n_e \left( \frac{m_e T}{2\pi} \right)^{-3/2} e^{B/T}, \quad (53)$$

between the number densities. Here $B = m_p + m_e - m_H = 13.6$ eV is the binding energy of hydrogen. The numbers of internal degrees of freedom are $g_p = g_e = 2$, $g_H = 4$. Outside the exponent we approximated $m_H \approx m_p$. Defining the fractional ionization

$$x = \frac{n_p}{n_B} \Rightarrow \frac{n_H}{n_p n_e} = \frac{(1 - x)}{x^2 n_B}. \quad (54)$$

Using (48), Eq. (53) becomes

$$\frac{1 - x}{x^2} = \frac{4\sqrt{2} \zeta(3)}{\sqrt{\pi}} \eta \left( \frac{T}{m_e} \right)^{3/2} e^{B/T}, \quad (55)$$

the Saha equation for ionization in thermal equilibrium. When $B \ll T \ll m_e$, the RHS $\ll 1$ so that $x \sim 1$, and almost all protons and electrons are free. As temperature falls, $e^{B/T}$ grows, but since both $\eta$ and $(T/m_e)^{3/2}$ are $\ll 1$, the temperature needs to fall to $T \ll B$, before the whole expression becomes large ($\sim 1$ or $\gg 1$).

The ionization fraction at first follows the equilibrium result of Eq. (55) closely, but as this equilibrium fraction begins to fall rapidly, the true ionization fraction begins to lag behind. As the number densities of free electrons and protons fall, it becomes more difficult for them to find each other to “recombine”, and they are no longer able to maintain chemical equilibrium for the reaction (52). To find the correct ionization evolution, $x(t)$, requires then a more complicated calculation involving the reaction cross section of this reaction. See Figs. 5 and 6.

Although the equilibrium formula is thus not enough to give us the true ionization evolution, its benefit is twofold:

1. It tells us when recombination begins. While the equilibrium ionization changes slowly, it is easy to stay in equilibrium. Thus things won’t start to happen until the equilibrium fraction begins to change a lot.

2. It gives the initial conditions for the more complicated calculation that will give the true evolution.

A similar situation holds for many other events in the early universe, e.g., big bang nucleosynthesis.

The recombination is not instantaneous. Let us define the recombination temperature $T_{rec}$ as the temperature where $x = 0.5$. Now $T_{rec} = T_0(1 + z_{rec})$ since $1 + z = a^{-1}$ and the photon temperature falls as $T \propto a^{-1}$. (Since $\eta \ll 1$, the energy release in recombination is negligible compared to $\rho_\gamma$; and after photon decoupling photons travel freely maintaining kinetic equilibrium with $T \propto a^{-1}$.)
Figure 5: Recombination. In the top panel the dashed curve gives the equilibrium ionization fraction as given by the Saha equation. The solid curve is the true ionization fraction, calculated using the actual reaction rates (original calculation by Peebles). You can see that the equilibrium fraction is followed at first, but then the true fraction lags behind. The bottom panel shows the free electron number density $n_e$ and the photon mean free path $\lambda_\gamma$. The latter is given in comoving units, i.e., the distance is scaled to the corresponding present distance. This figure is for $\eta = 8.22 \times 10^{-10}$. (Figure by R. Keskitalo.)

Figure 6: Same as Fig. 5, but with a logarithmic scale for the ionization fraction, and the time (actually redshift) scale extended to present time ($z = 0$ or $1 + z = 1$). You can see how a residual ionization $x \sim 10^{-4}$ remains. This figure does not include the reionization which happened at around $z \sim 10$. (Figure by R. Keskitalo.)
We get (for $\eta \sim 10^{-9}$)

\[
T_{\text{rec}} \sim 0.3 \text{ eV} \\
\eta_{\text{rec}} \sim 1300.
\]

You might have expected that $T_{\text{rec}} \sim B$. Instead we found $T_{\text{rec}} \ll B$. The main reason for this is that $\eta \ll 1$. This means that there are very many photons for each hydrogen atom. Even when $T \ll B$, the high-energy tail of the photon distribution contains photons with energy $E > B$ so that they can ionize a hydrogen atom.

The photon decoupling takes place somewhat later, at $T_{\text{dec}} \equiv (1+z_{\text{dec}})T_0$, when the ionization fraction has fallen enough. We define the photon decoupling time to be the time when the photon mean free path exceeds the Hubble distance. The numbers are roughly

\[
T_{\text{dec}} \sim 3000 \text{ K} \sim 0.26 \text{ eV} \\
\eta_{\text{dec}} \sim 1090.
\]

The decoupling means that the recombination reaction can not keep the ionization fraction on the equilibrium track, but instead we are left with a residual ionization of $x \sim 10^{-4}$.

A long time later ($z \sim 10$) the first stars form, and their radiation reionizes the gas that is left in interstellar space. The gas has now such a low density however, that the universe remains transparent.

---

**Exercise: Transparency of the universe.** We say the universe is transparent when the photon mean free path $\lambda_\gamma$ is larger than the Hubble length $l_H = H^{-1}$, and opaque when $\lambda_\gamma < l_H$. The photon mean free path is determined mainly by the scattering of photons by free electrons, so that $\lambda_\gamma = 1/(\sigma_T n_e)$, where $n_e = x n_e^* \eta$ is the number density of free electrons, $n_e^*$ is the total number density of electrons, and $x$ is the ionization fraction. The cross section for photon-electron scattering is independent of energy for $E_\gamma \ll m_e$, and is then called the Thomson cross section, $\sigma_T = \frac{4}{3} \pi (\alpha/m_e)^2$, where $\alpha$ is the fine-structure constant. In recombination $x$ falls from 1 to $10^{-4}$. Show that the universe is opaque before recombination and transparent after recombination. (Assume the recombination takes place between $z = 1300$ and $z = 1000$. You can assume a matter-dominated universe—see below for parameter values.) The interstellar matter gets later reionized (to $x \sim 1$) by the light from the first stars. What is the earliest redshift when this can happen without making the universe opaque again? (You can assume that most ($\sim$ all) matter has remained interstellar. Calculate for $\Omega_m = 1.0$ and $\Omega_m = 0.3$ (note that $\Omega_m$ includes nonbaryonic matter). Use $\Omega_\Lambda = 0$, $h = 0.7$ and $\eta = 6 \times 10^{-10}$.

The photons in the cosmic background radiation have thus traveled without scattering through space all the way since we had $T = T_{\text{dec}} = 1091 T_0$. When we look at this cosmic background radiation we thus see the universe (its faraway parts near our horizon) as it was at that early time. Because of the redshift, these photons which were then largely in the visible part of the spectrum, have now become microwave photons, so this radiation is now called the *cosmic microwave background* (CMB). It still maintains the kinetic equilibrium distribution. This was confirmed to high accuracy by the FIRAS (Far InfraRed Absolute Spectrophotometer) instrument on the COBE (Cosmic Background Explorer) satellite in 1989. John Mather received the 2006 Physics Nobel Prize for this measurement of the CMB frequency (photon energy) spectrum (see Fig. 7).\(^9\)

We shall now, for a while, stop the detailed discussion of the history of the universe at these events, recombination and photon decoupling. The universe is about 400 000 years old now. What will happen next, is that the structure of the universe (galaxies, stars) begins to form, as gravity begins to draw matter into overdense regions. Before photon decoupling the radiation pressure from photons prevented this. But before going to the physics of *structure formation* (discussed in Cosmology II) we shall discuss some earlier events (big bang nucleosynthesis, . . . ) in more detail.

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\(^9\)He shared the Nobel Prize with George Smoot, who got it for the discovery of the CMB anisotropy with the DMR instrument on the same satellite. The CMB anisotropy will be discussed in Cosmology II.
Figure 7: The CMB frequency spectrum as measured by the FIRAS instrument on the COBE satellite[3]. This first spectrum from FIRAS is based on just 9 minutes of measurements. The CMB temperature estimated from it was $T = 2.735 \pm 0.060 \text{ K}$. The final result[4] from FIRAS is $T = 2.725 \pm 0.001 \text{ K}$ (68% confidence; [4] actually gives this as $T = 2.725 \pm 0.002 \text{ K}$ at 95% confidence).

<table>
<thead>
<tr>
<th>Event</th>
<th>Temperature</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electroweak Transition</td>
<td>$T \sim 100 \text{ GeV}$</td>
<td>$t \sim 20 \text{ ps}$</td>
</tr>
<tr>
<td>QCD Transition</td>
<td>$T \sim 150 \text{ MeV}$</td>
<td>$t \sim 20 \mu\text{s}$</td>
</tr>
<tr>
<td>Neutrino Decoupling</td>
<td>$T \sim 1 \text{ MeV}$</td>
<td>$t \sim 1 \text{ s}$</td>
</tr>
<tr>
<td>Electron-Positron Annihilation</td>
<td>$T &lt; m_e \sim 0.5 \text{ MeV}$</td>
<td>$t \sim 10 \text{ s}$</td>
</tr>
<tr>
<td>Big Bang Nucleosynthesis</td>
<td>$T \sim 50$–$100 \text{ keV}$</td>
<td>$t \sim 10 \text{ min}$</td>
</tr>
<tr>
<td>Matter-Radiation Equality</td>
<td>$T \sim 0.8 \text{ eV} \sim 9000 \text{ K}$</td>
<td>$t \sim 60000 \text{ yr}$</td>
</tr>
<tr>
<td>Recombination + Photon Decoupling</td>
<td>$T \sim 0.3 \text{ eV} \sim 3000 \text{ K}$</td>
<td>$t \sim 380000 \text{ yr}$</td>
</tr>
</tbody>
</table>

Table 4: Early universe events.
4.9 The Dark Age

How would the universe after recombination appear to an observer with human eyes? At first one would see a uniform red glow everywhere, since the wavelengths of the CMB photons are in the visible range. (It would also feel rather hot, 3000 K). As time goes on this glow gets dimmer and dimmer as the photons redshift towards the infrared, and after a few million years it gets completely dark, as the photons become invisible infrared (heat) radiation. There are no stars yet. This is often called the dark age of the universe. It lasts several hundred million years. While it lasts, it gradually gets cold. In the dark, however, masses are gathering together. And then, one by one, the first stars light up.

The decoupling of photon from baryonic matter (electrons, protons, nuclei, ions, atoms) is actually very asymmetric, since there is over $10^9$ photons for each nucleus. The photon decoupling redshift $z = 1090$ is when photons decouple from baryons. After that, most photons will never scatter. However, some do, and these are enough to keep the temperature of the baryonic matter the same as photon temperature down to $z \sim 200$. After that, the decoupling is complete also from the baryonic point of view.\(^\text{10}\) The baryonic matter (mainly hydrogen and helium gas) remains in internal kinetic equilibrium, but its temperature $T_b$ falls now as $a^{-2}$ (momentum redshifts as $a^{-1}$, and for nonrelativistic particles kinetic energy is $p^2/2m$ and mean kinetic energy is related to temperature by $\langle E_k \rangle = 3T/2$). So at $z \sim 20$, the baryon temperature is only a few $K$, about 1/10 of the photon temperature then. This is their coldest moment, since sometime after $z \sim 20$ the first stars form and begin to heat up the interstellar gas.

It seems that the star-formation rate peaked between redshifts $z = 1$ and $z = 2$. Thus the universe at a few billion years was brighter than it is today, since the brightest stars are short-lived, and the galaxies were closer to each other then.\(^\text{11}\)

4.10 The radiation and neutrino backgrounds

While the starlight is more visible to us than the cosmic microwave background, its average energy density and photon number density in the universe is much less. Thus the photon density is essentially given by the CMB. The number density of CMB photons today ($T_0 = 2.725$ K) is

\[
 n_{\gamma 0} = \frac{2\zeta(3)}{\pi^2} T_0^3 = 410.5 \text{ photons/cm}^3
\]  

(56)

and the energy density is

\[
 \rho_{\gamma 0} = \frac{2\pi^2}{30} T_0^4 = 2.701 T_0 n_{\gamma 0} = 4.641 \times 10^{-31} \text{kg/m}^3.
\]  

(57)

Since the critical density is

\[
 \rho_{\text{cr}0} = \frac{3H_0^2}{8\pi G} = h^2 \cdot 1.8788 \times 10^{-26} \text{kg/m}^3
\]  

(58)

we get for the photon density parameter

\[
 \Omega_{\gamma} \equiv \frac{\rho_{\gamma 0}}{\rho_{\text{cr}0}} = 2.47 \times 10^{-5} h^{-2}.
\]  

(59)

\(^{10}\)There is a similar asymmetry in neutrino decoupling. From the neutrino point of view, the decoupling temperature is $T \sim 3$ MeV, from the baryonic point of view $T \sim 0.8$ MeV.

\(^{11}\)To be fair, galaxies seen from far away are rather faint objects, difficult to see with the unaided eye. In fact, if you were suddenly transported to a random location in the present universe, you might not be able to see anything. Thus, to enjoy the spectacle, our hypothetical observer should be located within a forming galaxy, or equipped with a good telescope.
While relativistic, neutrinos contribute another radiation component

\[ \rho_{\nu} = \frac{7N_{\nu}}{4} \frac{\pi^2}{30} T_{\nu}^4. \]  \hspace{1cm} (60)

After \( e^+ e^- \)-annihilation this gives

\[ \rho_{\nu} = \frac{7N_{\nu}}{8} \left( \frac{4}{11} \right)^4 \rho_{\gamma}, \]  \hspace{1cm} (61)

where \( N_{\nu} = 3 \) is the number of neutrino species.

When the number of neutrino species was not yet known, cosmology (BBN) was used to constrain it. Big bang nucleosynthesis is sensitive to the expansion rate in the early universe, and that depends on the energy density. Observations of abundances of light element isotopes combined with BBN calculations require \( N_{\nu} = 2-4 \). Actually any new light particle species that would be relativistic at nucleosynthesis time (\( T \sim 50 \text{ keV} - 1 \text{ MeV} \)) and would thus contribute to the expansion rate through its energy density, but which would not interact directly with nuclei and electrons, would have the same effect. Thus such hypothetical unknown particles (called \textit{dark radiation}) may not contribute to the energy density of the universe at that time more than one neutrino species does.

If we take Eq. (61) to define \( N_{\nu} \), but then take into account the extra contribution to \( \rho_{\nu} \) from energy leakage during \( e^+ e^- \)-annihilation (and some other small effects), we get (as a result of years of hard work by many theorists)

\[ N_{\nu} = 3.04. \]  \hspace{1cm} (62)

(So this does not mean that there are 3.04 neutrino species. It means that the total energy density in neutrinos is 3.04 times as much as the energy density one neutrino species would contribute had it decoupled completely before \( e^+ e^- \)-annihilation.)

If neutrinos are still relativistic today, the neutrino density parameter is

\[ \Omega_{\nu} = \frac{7N_{\nu}}{22} \left( \frac{4}{11} \right)^4 \Omega_{\gamma} = 1.71 \times 10^{-5} h^{-2}, \]  \hspace{1cm} (63)

so that the total radiation density parameter is

\[ \Omega_{\gamma} + \Omega_{\nu} = 4.18 \times 10^{-5} h^{-2} \sim 10^{-4}. \]  \hspace{1cm} (64)

We thus confirm the claim in Chapter 3, that the radiation component can be ignored in the Friedmann equation, except in the early universe. The combination \( \Omega_i h^2 \) is often denoted by \( \omega_i \), so we have

\[ \omega_{\gamma} = 2.47 \times 10^{-5} \]  \hspace{1cm} (65)
\[ \omega_{\nu} = 1.71 \times 10^{-5} \]  \hspace{1cm} (66)
\[ \omega_{\nu} = \omega_{\gamma} + \omega_{\nu} = 4.18 \times 10^{-5}. \]  \hspace{1cm} (67)

Neutrino oscillation experiments indicate a neutrino mass in the meV–eV range. This means that neutrinos are nonrelativistic today and count as matter, not radiation, except possibly the lightest neutrino species. Then the above result for the neutrino energy density of the present universe does not apply. However, unless the neutrino masses are above 0.2 eV, they would still have been relativistic, and counted as radiation, at the time of recombination and matter-radiation equality. While the neutrinos are relativistic, one still gets the neutrino energy density as

\[ \rho_{\nu} = \Omega_{\nu} \rho_{\text{cr0}} a^{-4} \]  \hspace{1cm} (68)
using the $\Omega_\nu$ of Eq. (63), even though this relation does not hold when the neutrinos become nonrelativistic and thus this $\Omega_\nu$ is not the density parameter to give the present density of neutrinos (we shall discuss that in Chapter 6).

**Exercise: Matter–radiation equality.** The present density of matter is $\rho_{m0} = \Omega_m \rho_{c0}$ and the present density of radiation is $\rho_{r0} = \rho_{r0} + \rho_{\nu0}$ (we assume neutrinos are massless). What was the age of the universe $t_{eq}$ when $\rho_m = \rho_r$? (Note that in these early times—but not today—you can ignore the curvature and vacuum terms in the Friedmann equation.) Give numerical value (in years) for the cases $\Omega_m = 0.1, 0.3, \text{and } 1.0$, and $H_0 = 70 \text{km/s/Mpc}$. What was the temperature ($T_{eq}$) then?

**References**


