9 Cosmic Microwave Background Anisotropy

9.1 Introduction

The cosmic microwave background (CMB) is isotropic to a high degree. This tells us that the early universe was rather homogeneous at the time \( t = t_{\text{dec}} \approx 380,000 \) years) the CMB was formed. However, with precise measurements we can detect a low-level anisotropy in the CMB (Fig. 1) which reflects the small perturbations in the early universe.

This anisotropy was first detected by the COBE (Cosmic Background Explorer) satellite in 1992, which mapped the whole sky in three microwave frequencies. The angular resolution of COBE was rather poor, \( 7^\circ \), meaning that only features larger than this were detected. Measurements with better resolution, but covering only small parts of the sky were then performed using instruments carried by balloons to the upper atmosphere, and ground-based detectors located at high altitudes. A significant improvement came with the WMAP (Wilkinson Microwave Anisotropy Probe) satellite, which made observations for nine years, from 2001 to 2010.

The best CMB anisotropy data to date, covering the whole sky, has been provided by the Planck satellite (Fig. 2). Planck was launched by the European Space Agency (ESA), on May 14th, 2009, to an orbit around the L2 point of the Sun-Earth system, 1.5 million kilometers from the Earth in the anti-Sun direction. Planck made observations for over four years, from August 12th, 2009 until October 23rd, 2013. The first major release of Planck results was in 2013 [1] and the second release in 2015 [2]. Final Planck results are expected in 2018.

Figure 1: Cosmic microwave background. The figure shows temperature variations from \(-400 \mu K\) to \(+400 \mu K\) around the mean temperature (2.725 K) over the whole sky, in galactic coordinates. The color is chosen to mimic the true color of CMB at the time it was formed, when it was visible orange-red light, but the brightness variation (the anisotropy) is hugely exaggerated by the choice of color scale. The fuzzy regions, notable especially in the galactic plane, are regions of the sky where microwave radiation from our own galaxy or nearby galaxies makes it difficult to separate out the CMB. (ESA/Planck data).

Planck observed the entire sky twice in a year. The satellite repeated these observations year after year, and the results become gradually more accurate, since the effects of instrument noise averaged out and various instrument-related systematic effects could be determined and corrected better with repeated observations.
Figure 2: The Planck satellite and its microwave receivers. The larger horns are for receiving lower frequencies and the smaller horns for higher frequencies.

Figure 3: Brightness of the sky in the nine Planck frequency bands. These sky maps are in galactic coordinates so the Milky Way lies horizontally. From [2].
In addition to the CMB, there is microwave radiation from our own galaxy and other galaxies, called *foreground* by those who study CMB. This radiation can be separated from the CMB based on its different electromagnetic spectrum. To enable this *component separation*, Planck observed at 9 different frequency bands; the lowest one centered at 30 GHz and the highest at 857 GHz (Fig. 3). There were two different instruments on Planck, using different technologies to detect the variations in the microwave radiation. The Low Frequency Instrument (LFI) used radiometers for the 30, 44, and 70 GHz bands. The High Frequency Instrument (HFI) used bolometers for the bands from 100 GHz to 857 GHz. HFI is the barrel-shaped instrument at the center in Fig. 2 right panel and LFI was wrapped around it. With the additional help of WMAP and ground-based data 8 different foreground components could be distinguished (Fig. 4).

![Figure 4: Result from Planck component separation. Also WMAP data and ground-based 408 MHz data was used. The extracted nine different components of the microwave radiation from top left to bottom right are: 1) CMB; 2) synchrotron radiation generated by relativistic cosmic-ray electrons accelerated by the galactic magnetic field; 3) “free-free emission” (bremsstrahlung) from electron-ion collisions; 4) emission from spinning galactic dust grains due to their electric dipole moment; 5) thermal emission from galactic dust (the typical dust temperatures are of order 20 K, so the dust thermal spectrum is peaked at much higher frequencies than CMB); 6) spectral line emission from HCN, CN, HCO, CS, and other molecules; 7) spectral line emission from the CO (carbon monoxide) $J = 1 \rightarrow 0$ transition; 8) CO $J = 2 \rightarrow 1$ line; 9) CO $J = 3 \rightarrow 2$ line (these emission lines from transitions between the four lowest rotation states of the CO molecule map the distribution of carbon monoxide in the Milky Way). From [2].]
Figures 5–7 show the observed variation $\delta T$ in the temperature of the CMB on the sky (red means hotter than average, blue means colder than average).

![Cosmic microwave background](image)

Figure 5: Cosmic microwave background: Fig. 1 reproduced in false color to bring out the patterns more clearly. The color range corresponds to CMB temperature variations from $-300 \mu K$ (blue) to $+300 \mu K$ (red) around the mean temperature. (ESA/Planck data).
Figure 6: The northern galactic hemisphere of the CMB sky (ESA/Planck data).
Figure 7: The southern galactic hemisphere of the CMB sky. The conspicuous cold region around \((-150^\circ, -55^\circ)\) is called the Cold Spot. The yellow smooth spot at \((-80^\circ, -35^\circ)\) in galactic coordinates is a region where the CMB is obscured by the Large Magellanic Cloud, and the light blue spot at \((-150^\circ, -20^\circ)\) is due to the Orion Nebula. (ESA/Planck data).
Figure 8: The observed CMB temperature anisotropy gets a contribution from the last scattering surface, 
\((\delta T/T)_{\text{intr}} = \Theta(t_{\text{dec}}, \mathbf{x}_{\text{ls}}, \mathbf{n})\) and from along the photon's journey to us, 
\((\delta T/T)_{\text{jour}}\).

The photons we see as the CMB, have traveled to us from where our past light cone intersects the hypersurface corresponding to the time \(t = t_{\text{dec}}\) of photon decoupling. This intersection forms a sphere which we shall call the last scattering surface.\(^1\) We are at the center of this sphere, except that timewise the sphere is located in the past.

The observed temperature anisotropy is due to two contributions, an intrinsic temperature variation at the surface of last scattering and a variation in the redshift the photons have suffered during their “journey” to us,

\[
\left( \frac{\delta T}{T} \right)_{\text{obs}} = \left( \frac{\delta T}{T} \right)_{\text{intr}} + \left( \frac{\delta T}{T} \right)_{\text{jour}}. \tag{1}
\]

See Fig. 8.

The first term, \((\delta T/T)_{\text{intr}}\) represents the temperature variation of the photon gas at \(t = t_{\text{dec}}\). We also include in it the Doppler effect from the motion of this photon gas. At that time the larger scales we see in the CMB sky were still outside the horizon, so we have to pay attention to the gauge choice. In fact, the separation of \(\delta T/T\) into the two components in Eq. (1) is gauge-dependent. If the time slice \(t = t_{\text{dec}}\) dips further into the past in some location, it finds a higher temperature, but the photons from there also have a longer way to go and suffer a larger redshift, so that the two effects balance each other. We can calculate in any gauge we want, getting different results for \((\delta T/T)_{\text{intr}}\) and \((\delta T/T)_{\text{jour}}\) depending on the gauge, but their sum \((\delta T/T)_{\text{obs}}\) is gauge independent. It has to be, being an observed quantity.

One might think that \((\delta T/T)_{\text{intr}}\) should be equal to zero, since in our earlier discussion of recombination and decoupling we identified decoupling with a particular temperature \(T_{\text{dec}} \sim 3000\text{ K}\). This kind of thinking corresponds to a particular gauge choice where the \(t = t_{\text{dec}}\) time slice coincides with the \(T = T_{\text{dec}}\) hypersurface. In this gauge \((\delta T/T)_{\text{intr}} = 0\), except for the Doppler effect (we are not going to use this gauge). Anyway, it is not true that all photons have their last scattering exactly when \(T = T_{\text{dec}}\). Rather they occur during a rather large temperature interval and time period. The zeroth-order (background) time evolution of the temperature of the photon distribution is the same before and after last scattering, \(T \propto a^{-1}\), so it does not matter how we draw the artificial separation line, the time slice \(t = t_{\text{dec}}\) separating the fluid and free particle treatments of the photons. See Fig. 9.

\(^1\)Or the last scattering sphere. “Last scattering surface” often refers to the entire \(t = t_{\text{dec}}\) time slice.
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Figure 9: Depending on the gauge, the $T = T_{\text{dec}}$ surface may, or (usually) may not coincide with the $t = t_{\text{dec}}$ time slice.

9.2 Multipole analysis

The CMB temperature anisotropy is a function over a sphere (the celestial sphere, or the unit sphere of directions $\mathbf{n}$). In analogy with the Fourier expansion in 3D space, we separate out the contributions of different angular scales by doing a multipole expansion,

$$\frac{\delta T}{T_0}(\theta, \phi) = \sum a_{\ell m} Y_{\ell m}(\theta, \phi)$$

where the sum runs over $\ell = 1, 2, \ldots \infty$ and $m = -\ell, \ldots, \ell$, giving $2\ell + 1$ values of $m$ for each $\ell$. The functions $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics (see Fig. 10), which form an orthonormal set of functions over the sphere, so that we can calculate the multipole coefficients $a_{\ell m}$ from

$$a_{\ell m} = \int Y_{\ell m}^* (\theta, \phi) \frac{\delta T}{T_0} (\theta, \phi) d\Omega.$$

Definition (2) gives dimensionless $a_{\ell m}$. Often they are defined without the $T_0 = 2.725 \text{ K}$ in Eq. (2), and then they have the dimension of temperature and are usually given in units of $\mu\text{K}$.

Here $\theta$ and $\phi$ are spherical coordinates, $d\Omega \equiv d\cos \theta d\phi$, $\theta$ ranges from 0 to $\pi$ and $\phi$ ranges from 0 to $2\pi$.\footnote{They can also be given in degrees, the colatitude $\theta$ ranging from 0° (North) to 180° (South) and the longitude $\phi$ from 0° to 360°. There are a number of different astronomical coordinate systems (equatorial, ecliptic, galactic) in use, with their own historical conventions for the coordinate names, symbols, and units. Typically they involve the latitude $90^\circ - \theta$ instead of the colatitude, so that North is at $+90^\circ$ and South at $-90^\circ$, and the longitude is usually given between $-180^\circ$ and $+180^\circ$, e.g., in Fig. 7.}

The sum begins at $\ell = 1$, since $Y_{00} = \text{const.}$ and therefore we must have $a_{00} = 0$ for a quantity which represents a deviation from average. The dipole part, $\ell = 1$, is dominated by the Doppler effect due to the motion of the solar system with respect to the last scattering surface, and we cannot separate out from it the cosmological dipole caused by large scale perturbations. Therefore we are here interested only in the $\ell \geq 2$ part of the expansion.

Another notation for $Y_{\ell m}(\theta, \phi)$ is $Y_{\ell m}(\mathbf{n})$, where $\mathbf{n}$ is a unit vector whose direction is specified by the angles $\theta$ and $\phi$.

9.2.1 Spherical harmonics

We list here some useful properties of the spherical harmonics.

They are orthonormal functions on the sphere, so that

$$\int d\Omega \ Y_{\ell m}(\theta, \phi) Y_{\ell' m'}^*(\theta, \phi) = \delta_{\ell \ell'} \delta_{m m'}.$$

They are elementary complex functions and are related to the associated Legendre functions $P_{\ell}^m(x)$ by

$$Y_{\ell m}(\theta, \phi) = (-1)^m \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_{\ell}^m(\cos \theta) e^{im\phi}.$$
Legendre polynomials

\[ P_0(x) = 1 \]
\[ P_1(x) = x \]
\[ P_2(x) = \frac{1}{2}(3x^2 - 1) \]
\[ P_3(x) = \frac{1}{2}(5x^3 - 3x) \]
\[ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \]

Associated Legendre functions \( P^m_\ell(x) = P^m_\ell(\cos \theta) \)

\[ P^1_1(x) = \sqrt{1 - x^2} = \sin \theta \]
\[ P^2_1(x) = 3x\sqrt{1 - x^2} = 3 \cos \theta \sin \theta \]
\[ P^2_2(x) = 3(1 - x^2) = 3 \sin^2 \theta \]

Spherical harmonics

\[ Y^0_\ell(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \]
\[ Y^1_\ell(\theta, \phi) = -\sqrt{\frac{2}{\pi}} \sin \theta e^{i\phi} \]
\[ Y^2_\ell(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta \]
\[ Y^3_\ell(\theta, \phi) = \sqrt{\frac{1}{24\pi}} 3 \sin^2 \theta e^{i2\phi} \]
\[ Y^4_\ell(\theta, \phi) = \sqrt{\frac{1}{4\pi}} (\frac{3}{2} \cos^2 \theta - \frac{1}{2}) \]

Spherical Bessel functions

\[ j_0(x) = \frac{\sin x}{x} \]
\[ j_1(x) = \frac{\sin x}{x} - \frac{\cos x}{x} \]
\[ j_2(x) = \left( \frac{3}{x^2} - \frac{1}{x} \right) \sin x - \frac{3}{x^2} \cos x \]

Table 1: Legendre functions, spherical harmonics, and spherical Bessel functions.

Thus the \( \theta \)-dependence is in \( P^m_\ell(\cos \theta) \) and the \( \phi \)-dependence is in \( e^{im\phi} \). The functions \( P^m_\ell \) are real and

\[ Y_{\ell,-m} = (-1)^m Y^{*}_{\ell m}, \quad (6) \]

so that

\[ Y_{00} = \sqrt{\frac{2\ell + 1}{4\pi}} P_{\ell}(\cos \theta) \quad \text{is real.} \quad (7) \]

The functions \( P_{\ell} \equiv P^{0}_{\ell} \) are called Legendre polynomials. See Table 9.2.1 for examples of these functions for \( \ell \leq 2 \).

Summing over the \( m \) corresponding to the same multipole number \( \ell \) gives the addition theorem

\[ \sum_m Y^{*}_{\ell m}(\theta', \phi')Y_{\ell m}(\theta, \phi) = \frac{2\ell + 1}{4\pi} P_{\ell}(\cos \theta), \quad (8) \]
where \( \vartheta \) is the angle between \( \mathbf{n} = (\theta, \phi) \) and \( \mathbf{n}' = (\theta', \phi') \), i.e., \( \mathbf{n} \cdot \mathbf{n}' = \cos \vartheta \). For \( \mathbf{n} = \mathbf{n}' \) this becomes

\[
\sum_m |Y_{\ell m}(\theta, \phi)|^2 = \frac{2\ell + 1}{4\pi} \tag{9}
\]

(since \( P_{\ell}(1) = 1 \) always).

We shall also need the expansion of a plane wave in terms of spherical harmonics,

\[
e^{i \mathbf{k} \cdot \mathbf{x}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kx)Y_{\ell m}(\hat{x})Y^*_{\ell m}(\hat{k}) \tag{10}
\]

Here \( \hat{x} \) and \( \hat{k} \) are the unit vectors in the directions of \( x \) and \( k \), and the \( j_\ell \) are the spherical Bessel functions.

### 9.2.2 Theoretical angular power spectrum

The CMB anisotropy is due to primordial perturbations, and therefore it reflects their Gaussian nature. Because one gets the values of the \( a_{\ell m} \) from the other perturbation quantities through linear equations (in first-order perturbation theory), the \( a_{\ell m} \) are also (complex) Gaussian random variables. Since they represent a deviation from the average temperature, their expectation value is zero,

\[
\langle a_{\ell m} \rangle = 0 \tag{11}
\]

From statistical isotropy follows that the \( a_{\ell m} \) are independent random variables so that

\[
\langle a_{\ell m} a_{\ell' m'}^* \rangle = 0 \quad \text{if } \ell \neq \ell' \text{ or } m \neq m' \tag{12}
\]

Since \( \delta T/T_0 \) is real,

\[
a_{\ell, -m} = (-1)^m a_{\ell m}^* \tag{13}
\]

Although thus \( a_{\ell, -m} \) and \( a_{\ell m} \) are not independent of each other, we still have \( \langle a_{\ell m} a_{\ell', -m'}^* \rangle = 0 \) (exercise), so that (12) is satisfied even in this case. For each \( \ell \), there are \( 2\ell + 1 \) independent real random variables: \( a_{\ell 0} \) (which is always real), and \( \text{Re} a_{\ell m} \) and \( \text{Im} a_{\ell m} \) for \( m = 1, \ldots, \ell \).

The quantity we want to calculate from theory is the variance \( \langle |a_{\ell m}|^2 \rangle \) to get a prediction for the typical size of the \( a_{\ell m} \). From statistical isotropy also follows that these expectation values depend only on \( \ell \) not \( m \). (The \( \ell \) are related to the angular size of the anisotropy pattern, whereas the \( m \) are related to “orientation” or “pattern”. See Fig. 10.) Since \( \langle |a_{\ell m}|^2 \rangle \) is independent of \( m \), we can define

\[
C_\ell \equiv \langle |a_{\ell m}|^2 \rangle = \frac{1}{2\ell + 1} \sum_m \langle |a_{\ell m}|^2 \rangle \tag{14}
\]

and altogether we have

\[
\langle a_{\ell m} a_{\ell' m'}^* \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell \tag{15}
\]

This function \( C_\ell \) (of integers \( \ell \geq 2 \)) is called the (theoretical) angular power spectrum. It is analogous to the power spectrum \( P(k) \) of density perturbations. For Gaussian perturbations, the \( C_\ell \) contains all the statistical information about the CMB temperature anisotropy. And this is all we can predict from theory. Thus the analysis of the CMB anisotropy consists of calculating the angular power spectrum from the observed CMB (a map like Figure 5) and comparing it to the \( C_\ell \) predicted by theory.\(^3\)

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\(^3\)In addition to the temperature anisotropy, the CMB also has another property, its polarization. There are two additional power spectra related to the polarization, \( C_\ell^{EE} \) and \( C_\ell^{BB} \), and one related to the correlation between temperature and polarization, \( C_\ell^{TE} \).
Figure 10: The three lowest multipoles $\ell = 1, 2, 3$ of spherical harmonics. Left column: $Y_{10}$, $\text{Re}Y_{11}$, $\text{Im}Y_{11}$. Middle column: $Y_{20}$, $\text{Re}Y_{21}$, $\text{Im}Y_{21}$, $\text{Re}Y_{22}$, $\text{Im}Y_{22}$. Right column: $Y_{30}$, $\text{Re}Y_{31}$, $\text{Im}Y_{31}$, $\text{Re}Y_{32}$, $\text{Im}Y_{32}$, $\text{Re}Y_{33}$, $\text{Im}Y_{33}$. Figure by Ville Heikkilä.
Just like the 3D density power spectrum $P(k)$ gives the contribution of scale $k$ to the density variance $\langle \delta(x)^2 \rangle$, the angular power spectrum $C_\ell$ is related to the contribution of multipole $\ell$ to the temperature variance,

$$\left\langle \left( \frac{\delta T(\theta, \phi)}{T} \right)^2 \right\rangle = \left\langle \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi) \sum_{\ell' m'} a^*_{\ell' m'} Y^*_{\ell' m'}(\theta, \phi) \right\rangle$$

$$= \sum_{\ell m} Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta, \phi) \langle a_{\ell m} a^*_{\ell m} \rangle$$

$$= \sum_\ell C_\ell \sum_m |Y_{\ell m}(\theta, \phi)|^2 = \sum_\ell \frac{2\ell + 1}{4\pi} C_\ell,$$  \hspace{1cm} (16)

where we used (15) and (9). Thus, if we plot $(2\ell + 1)C_\ell/4\pi$ on a linear $\ell$ scale, or $\ell(2\ell + 1)C_\ell/4\pi$ on a logarithmic $\ell$ scale, the area under the curve gives the temperature variance, i.e., the expectation value for the squared deviation from the average temperature. It has become customary to plot the angular power spectrum as $\ell(2\ell + 1)C_\ell/2\pi$, which is neither of these, but for large $\ell$ approximates the second case. The reason for this custom is explained later.

Equation (16) represents the expectation value from theory and thus it is the same for all directions $\theta, \phi$. The actual, “realized”, value of course varies from one direction $\theta, \phi$ to another.

We can imagine an ensemble of universes, otherwise like our own, but representing a different realization of the same random process of producing the primordial perturbations. Then $\langle \rangle$ represents the average over such an ensemble.

Equation (16) can be generalized to the angular correlation function \textbf{(exercise)}

$$C(\vartheta) \equiv \left\langle \frac{\delta T(\mathbf{n}) \delta T(\mathbf{n}')}{T} \right\rangle = \frac{1}{4\pi} \sum_\ell (2\ell + 1)C_\ell \delta_{\ell \ell'} \delta_{mm'}, \hspace{1cm} (17)$$

where $\vartheta$ is the angle between $\mathbf{n}$ and $\mathbf{n}'$.

\textbf{9.2.3 Observed angular power spectrum}

Theory predicts expectation values $\langle |a_{\ell m}|^2 \rangle$ from the random process responsible for the CMB anisotropy, but we can observe only one realization of this random process, the set $\{a_{\ell m}\}$ of our CMB sky. We define the \textit{observed} angular power spectrum as the average

$$\hat{C}_\ell = \frac{1}{2\ell + 1} \sum_m |a_{\ell m}|^2 \hspace{1cm} (18)$$

of these observed values.

The variance of the observed temperature anisotropy is the average of $\left( \frac{\delta T(\theta, \phi)}{T} \right)^2$ over the celestial sphere,

$$\frac{1}{4\pi} \int \left[ \frac{\delta T(\theta, \phi)}{T} \right]^2 d\Omega = \frac{1}{4\pi} \int d\Omega \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi) \sum_{\ell' m'} a^*_{\ell' m'} Y^*_{\ell' m'}(\theta, \phi)$$

$$= \frac{1}{4\pi} \sum_{\ell m} \sum_{\ell' m'} a_{\ell m} a^*_{\ell' m'} \int Y_{\ell m}(\theta, \phi) Y^*_{\ell' m'}(\theta, \phi) d\Omega$$

$$= \frac{1}{4\pi} \sum_\ell \sum_m |a_{\ell m}|^2 = \sum_\ell \frac{2\ell + 1}{4\pi} C_\ell \hspace{1cm} (19)$$
Figure 11: The angular power spectrum $\hat{C}_\ell$ as observed by Planck. The observational results are the red data points with small error bars. The green curve is the theoretical $C_\ell$ from a best-fit model, and the light green band around it represents the cosmic variance corresponding to this $C_\ell$. The quantity plotted is actually $D_\ell \equiv T^2_\ell [\ell(\ell+1)/(2\pi)] C_\ell$. Note that the $\ell$-axis is logarithmic until 50 and linear after that. (This is Fig. 21 of [1].)

Contrast this with (16), which gives the variance of $\delta T/T$ at an arbitrary location on the sky over different realizations of the random process which produced the primordial perturbations; whereas (19) gives the variance of $\delta T/T$ of our given sky over the celestial sphere.

9.2.4 Cosmic Variance

The expectation value of the observed spectrum $\hat{C}_\ell$ is equal to $C_\ell$, the theoretical spectrum of Eq. (14), i.e.,

$$\langle \hat{C}_\ell \rangle = C_\ell \Rightarrow \langle \hat{C}_\ell - C_\ell \rangle = 0,$$

but its actual, realized, value is not, although we expect it to be close. The expected squared difference between $\hat{C}_\ell$ and $C_\ell$ is called the cosmic variance. We can calculate it using the properties of (complex) Gaussian random variables (exercise). The answer is

$$\langle (\hat{C}_\ell - C_\ell)^2 \rangle = \frac{2}{2\ell + 1} C_\ell^2.$$

We see that the expected relative difference between $\hat{C}_\ell$ and $C_\ell$ is smaller for higher $\ell$. This is because we have a larger (size $2\ell + 1$) statistical sample of $a_{\ell m}$ available for calculating the $\hat{C}_\ell$.

The cosmic variance limits the accuracy of comparison of CMB observations with theory, especially for large scales (low $\ell$). See Fig. 11.

9.3 Multipoles and scales

9.3.1 Rough correspondence

The different multipole numbers $\ell$ correspond to different angular scales, low $\ell$ to large scales and high $\ell$ to small scales. Examination of the functions $Y_{\ell m}(\theta, \phi)$ reveals that they have an
oscillatory pattern on the sphere, so that there are typically \( \ell \) “wavelengths” of oscillation around a full great circle of the sphere. See Figs. 10 and 12.

Thus the angle corresponding to this wavelength is

\[
\vartheta_{\lambda} = \frac{2\pi}{\ell} = \frac{360^\circ}{\ell}.
\]

See Fig. 13. The angle corresponding to a “half-wavelength”, i.e., the separation between a neighboring minimum and maximum is then

\[
\vartheta_{\text{res}} = \frac{\pi}{\ell} = \frac{180^\circ}{\ell}.
\]

This is the angular resolution required of the microwave detector for it to be able to resolve the angular power spectrum up to this \( \ell \).

For example, COBE had an angular resolution of 7\(^\circ\) allowing a measurement up to \( \ell = 180/7 = 26 \), WMAP had resolution 0.23\(^\circ\) reaching to \( \ell = 180/0.23 = 783 \), and Planck had resolution 5\('\), allowing the measurement of \( C_\ell \) up to \( \ell = 2160 \).

The angles on the sky are related to actual physical distances via the angular diameter distance \( d_A \), defined as the ratio of the physical length (transverse to the line of sight) and the angle it covers (see Chapter 3),

\[
d_A \equiv \frac{\lambda_{\text{phys}}}{\vartheta}.
\]

Likewise, we defined the comoving angular diameter distance \( d_A^c \) by

\[
d_A^c \equiv \frac{\lambda^c}{\vartheta}.
\]

\(^4\)In reality, there is no sharp cut-off at a particular \( \ell \), the observational error bars just blow up rapidly around this value of \( \ell \).
where $\lambda^c = a^{-1}\lambda_{\text{phys}} = (1 + z)\lambda_{\text{phys}}$ is the corresponding comoving length. Thus $d_A^c = a^{-1}d_A = (1 + z)d_A$. See Fig. 14.

Consider now the Fourier modes of our earlier perturbation theory discussion. A mode with comoving wavenumber $k$ has comoving wavelength $\lambda^c = 2\pi/k$. Thus this mode should show up as a pattern on the CMB sky with angular size

$$\vartheta = \frac{\lambda^c}{d_A^c} = \frac{2\pi}{kd_A^c} = \frac{2\pi}{\ell}. \quad (26)$$

For the last equality we used the relation (22). From it we get that the modes with wavenumber $k$ contribute mostly to multipoles around

$$\ell = kd_A^c. \quad (27)$$

### 9.3.2 Exact treatment

The above matching of wavenumbers with multipoles was of course rather naive, for two reasons:

1. The description of a spherical harmonic $Y_{\ell m}$ having an “angular wavelength” of $2\pi/\ell$ is just a crude characterization. See Fig. 12.

2. The modes $k$ are not wrapped around the sphere of last scattering, but the wave vector forms a different angle with the sphere at different places.
The following precise discussion applies only for the case of a flat universe \((K = 0\) Friedmann model as the background), where one can Fourier expand functions on a time slice. We start from the expansion of the plane wave in terms of spherical harmonics, for which we have the result, Eq. (10),

\[
e^{ik \cdot x} = 4\pi \sum_{\ell m} \ell j_{\ell}(kx) Y_{\ell m}(\hat{x}) Y_{\ell m}^*(\hat{k}) ,
\]

(28)

where \(j_{\ell}\) is the spherical Bessel function.

Consider now some function

\[
f(x) = \sum_{k} f_{k} e^{ik \cdot x}
\]

(29)
on the \(t = t_{\text{dec}}\) time slice. We want the multipole expansion of the values of this function on the last scattering sphere. See Fig. 15. These are the values \(f(x \hat{x})\), where \(x \equiv |x|\) has a constant value, the (comoving) radius of this sphere. Thus

\[
a_{\ell m} = \int d\Omega_x Y_{\ell m}^*(\hat{x}) f(x \hat{x})
\]

\[
= \sum_{k} \int d\Omega_x Y_{\ell m}^*(\hat{x}) f_{k} e^{ik \cdot x}
\]

\[
= 4\pi \sum_{k} \sum_{\ell m'} \int d\Omega_x f_{k} Y_{\ell m}(\hat{x}) i^{\ell} j_{\ell}(kx) Y_{\ell m'}^*(\hat{k})
\]

\[
= 4\pi i^{\ell} \sum_{k} f_{k} j_{\ell}(kx) Y_{\ell m}^*(\hat{k}),
\]

(30)

where we used the orthonormality of the spherical harmonics. The corresponding result for a Fourier transform \(f(k)\) is

\[
a_{\ell m} = \frac{4\pi i^{\ell}}{(2\pi)^3} \int d^3 k f(k) j_{\ell}(kx) Y_{\ell m}^*(\hat{k}).
\]

(31)

The \(j_{\ell}\) are oscillating functions with decreasing amplitude. For large values of \(\ell\) the position of the first (and largest) maximum is near \(kx = \ell\) (see Fig. 16).
Thus the $a_{\ell m}$ pick a large contribution from those Fourier modes $k$ where

$$kx \sim \ell .$$

(32)

In a flat universe the comoving distance $x$ (from our location to the sphere of last scattering) and the comoving angular diameter distance $d_c^\varphi$ are equal, so we can write this result as

$$kd_c^\varphi \sim \ell .$$

(33)

The conclusion is that a given multipole $\ell$ acquires a contribution from modes with a range of wavenumbers, but most of the contribution comes from near the value given by Eq. (27). This concentration is tighter for larger $\ell$.

We shall use Eq. (27) for qualitative purposes in the following discussion.

9.4 Important distance scales on the last scattering surface

9.4.1 Angular diameter distance to last scattering

In Chapter 3 we derived the formula for the comoving distance to redshift $z$,

$$d_c^\varphi (z) = H_0^{-1} \int_{\frac{1}{1+z}}^1 \frac{da}{\sqrt{\Omega_0 (a-a^2) - \Omega_\Lambda (a-a^4) + a^2}}$$

(34)

(where we have approximated $\Omega_0 \approx \Omega_m + \Omega_\Lambda$) and the corresponding comoving angular diameter distance

$$d_A^\varphi (z) = f_K (d_c^\varphi (z)) ,$$

(35)
where

\[ f_K(x) \equiv \begin{cases} 
K^{-1/2} \sin(K^{1/2}x), & K > 0 \\
x, & K = 0 \\
|K|^{-1/2} \sinh(|K|^{1/2}x), & K < 0
\end{cases} \quad (36) \]

We also define

\[ f_k(x) \equiv \begin{cases} 
\sin x, & k = 1 \\
x, & k = 0 \\
\sinh x, & k = -1.
\end{cases} \quad (37) \]

For the flat universe \((K = k = 0, \Omega_0 = 1)\), the comoving angular diameter distance is equal to the comoving distance,

\[ d_A^c(z) = d^c(z) \quad (K = 0). \quad (38) \]

For the open \((K < 0, \Omega_0 < 1)\) and closed \((K > 0, \Omega_0 > 1)\) cases we can write Eq. (35) as

\[
d_A^c(z) = \frac{H_0^{-1}}{\sqrt{\Omega_k}} f_k \left( \frac{\sqrt{\Omega_k}}{H_0^{-1}} d^c(z) \right) \\
= H_0^{-1} \frac{1}{\sqrt{\Omega_k}} f_k \left( \sqrt{\Omega_k} \int_{\frac{1}{1+z}}^{1} \frac{da}{\sqrt{\Omega_0(a-a^2) - \Omega_A(a-a^4) + a^2}} \right). \quad (39)\]

Thus \(d_A^c(z) \propto H_0^{-1}\), and has some more complicated dependence on \(\Omega_0\) and \(\Omega_A\) (or on \(\Omega_m\) and \(\Omega_\Lambda\)).

We are now interested in the distance to the last scattering sphere, i.e., \(d_A^c(z_{\text{dec}})\), where \(z_{\text{dec}} \approx 1090\).

For the simplest case, \(\Omega_A = 0, \Omega_m = 1\), the integral gives

\[ d_A^c(z_{\text{dec}}) = H_0^{-1} \int_{\frac{1}{1+z}}^{1} \frac{dx}{\sqrt{x}} = 2H_0^{-1} \left( 1 - \frac{1}{\sqrt{1+z_{\text{dec}}}} \right) = 1.94H_0^{-1} \approx 2H_0^{-1}, \quad (40) \]

where the last approximation corresponds to ignoring the contribution from the lower limit.

We shall consider two more general cases, of which the above is a special case of both:

a) Open universe with no dark energy: \(\Omega_A = 0\) and \(\Omega_m = \Omega_0 < 1\). Now the integral gives

\[
d_A^c(z_{\text{dec}}) = \frac{H_0^{-1}}{\sqrt{1-\Omega_m}} \sinh \left( \sqrt{1-\Omega_m} \int_{\frac{1}{1+z}}^{1} \frac{dx}{\sqrt{(1-\Omega_m)x^2 + \Omega_m x}} \right) \\
= \frac{H_0^{-1}}{\sqrt{1-\Omega_m}} \sinh \left( \int_{\frac{1}{1+z}}^{1} \frac{dx}{\sqrt{x^2 + \Omega_m x}} \right) \\
= \frac{H_0^{-1}}{\sqrt{1-\Omega_m}} \sinh \left( 2 \arcsinh \frac{1-\Omega_m}{\Omega_m} - 2 \arcsinh \frac{1-\Omega_m}{\Omega_m} \frac{1}{1+z_{\text{dec}}} \right) \\
\approx \frac{H_0^{-1}}{\Omega_m} \frac{2 \arcsinh \frac{1-\Omega_m}{\Omega_m}}{\Omega_m} = 2H_0^{-1}, \quad (41)\]

where again the approximation ignores the contribution from the lower limit (i.e., it actually gives the angular diameter distance to the horizon, \(d_A^c(z = \infty)\), in a model where we ignore the effect of other energy density components besides matter). In the last step we used \(\sinh 2x = 2 \sinh x \cosh x = 2 \sinh x \sqrt{1 + \sinh^2 x}\). We show this result (together with \(d^c(z = \infty)\)) in Fig. 17.
b) Flat universe with vacuum energy, $\Omega_\Lambda + \Omega_m = 1$. Here the integral does not give an elementary function, but a reasonable approximation, which we shall use in the following, is

$$d_A^c(z_{\text{dec}}) \approx \frac{2}{\Omega_m^{0.4} H_0^{-1}}. \quad (42)$$

The comoving distance $d_c(z_{\text{dec}})$ depends on the expansion history of the universe. The longer it takes for the universe to cool from $T_{\text{dec}}$ to $T_0$ (i.e., to expand by the factor $1 + z_{\text{dec}}$), the longer distance the photons have time to travel. When a larger part of this time is spent at small values of the scale factor, this distance gets a bigger boost from converting it to a comoving distance. For open/closed universes the angular diameter distance gets an additional effect from the geometry of the universe (the $f_K$), which acts like a “lens” to make the distant CMB pattern at the last scattering sphere to look smaller or larger (see Fig. 18).

### 9.4.2 Hubble scale and the matter-radiation equality scale

Subhorizon ($k \gg H$) and superhorizon ($k \ll H$) scales behave differently. Thus we want to know which of the structures we see on the last scattering surface are subhorizon and which are superhorizon. For that we need to know the comoving Hubble scale $H$ at $t_{\text{dec}}$. This was discussed in Sec. 8.3.1. At that time both matter and radiation are contributing to the energy density and the Hubble parameter. The scale which is just entering at $t = t_{\text{dec}}$ is

$$k_{\text{dec}}^{-1} \equiv \mathcal{H}_{\text{dec}}^{-1} = (1 + z_{\text{dec}}) H_0^{-1} = (1 + z_{\text{dec}})^{-1/2} H_0^{-1} \Omega_m^{-1/2} \left[ 1 + \frac{\Omega_r}{\Omega_m} (1 + z_{\text{dec}}) \right]^{-1/2} = \Omega_m^{-1/2} (1 + 0.046 \omega_m^{-1})^{-1/2} 91 h^{-1} \text{Mpc} \quad (43)$$
Figure 18: The geometry effect in a closed (top) or an open (bottom) universe affects the angle at which we see a structure of given size at the last scattering surface, and thus its angular diameter distance.

(using $z_{\text{dec}} = 1090$; here $0.046 \omega_m^{-1}$ is $\rho_r/\rho_m$ at $t_{\text{dec}}$) and the corresponding multipole number on the last scattering sphere is

$$l_H = k_{\text{dec}} d_A^0 = \frac{1}{2} \frac{1}{2} \Omega_m^{-1/2} \left[ 1 + \frac{\Omega_r}{\Omega_m} (1 + z_{\text{dec}}) \right]^{1/2} \left\{ \begin{array}{l} \frac{2}{\Omega_m} = 66 \Omega_m^{-0.5} \sqrt{1 + 0.046 \omega_m^{-1}} \quad (\Omega_\Lambda = 0) \\ \frac{2}{\Omega_m^{0.4}} \approx 66 \Omega_m^{0.1} \sqrt{1 + 0.046 \omega_m^{-1}} \quad (\Omega_0 = 1) \end{array} \right.$$ (44)

The angle subtended by a half-wavelength $\pi/k$ of this mode on the last scattering sphere is

$$\vartheta_H = \frac{\pi}{l_H} = \frac{180^\circ}{l_H} = \sqrt{1 + 0.046 \omega_m^{-1}} \times \left\{ \begin{array}{l} \frac{2.7^\circ \Omega_m^{0.5}}{2.7^\circ \Omega_m^{0.1}} \quad (\Omega_\Lambda = 0) \\ \frac{2.7^\circ \Omega_m^{0.5}}{2.7^\circ \Omega_m^{0.1}} \quad (\Omega_0 = 1) \end{array} \right.$$ (45)

For $\Omega_m \sim 0.3$, $\Omega_\Lambda \sim 0.7$, $h \sim 0.7$, $\ell_H \approx 67$ and $\vartheta_H \approx 3.5^\circ$.

Another important scale is $k_{\text{eq}}$, the scale which enters at the time of matter-radiation equality $t_{\text{eq}}$, since the transfer function $T(k)$ is bent at that point. Perturbations for scales $k \ll k_{\text{eq}}$ maintain essentially their primordial spectrum, whereas scales $k \gg k_{\text{eq}}$ have lost relative power between their horizon entry and $t_{\text{eq}}$. This scale is

$$k_{\text{eq}}^{-1} = H_{\text{eq}}^{-1} \approx 13.7 \Omega_m^{-1} h^{-2} \text{Mpc} = 4.6 \times 10^{-3} \Omega_m^{-1} h^{-1} H_0^{-1}$$ (46)

and the corresponding multipole number of these scales seen on the last scattering sphere is

$$l_{\text{eq}} = k_{\text{eq}} d_A^0 = 219 \Omega_m h \times \left\{ \begin{array}{l} \frac{2}{\Omega_m} = 440 h \quad (\Omega_\Lambda = 0) \\ \frac{2}{\Omega_m^{0.4}} \approx 440 h \Omega_m^{0.6} \quad (\Omega_0 = 1) \end{array} \right.$$ (47)

9.5 CMB anisotropy from perturbation theory

We began this chapter with the observation, Eq. (1), that the CMB temperature anisotropy is a sum of two parts,

$$\left( \frac{\delta T}{T} \right)_{\text{obs}} = \left( \frac{\delta T}{T} \right)_{\text{intr}} + \left( \frac{\delta T}{T} \right)_{\text{jour}},$$ (48)
and that this separation is gauge dependent. We shall consider this in the conformal-Newtonian gauge, since the second part, \((\Psi/T)_\text{jour}\), the integrated redshift perturbation along the line of sight, is easiest to calculate in this gauge. (However, we won’t do the calculation here.\(^5\))

The result of this calculation is
\[
\left(\frac{\delta T}{T}\right)_\text{jour} = -\int d\Phi + \int (\dot{\Phi} + \dot{\Psi}) dt + \mathbf{v}_{\text{obs}} \cdot \hat{n}
\]
\[
= \Phi(t_{\text{dec}}, \mathbf{x}_{\text{ls}}) - \Phi(t_0, 0) + \int (\dot{\Phi} + \dot{\Psi}) dt + \mathbf{v}_{\text{obs}} \cdot \hat{n}
\]
\[
\approx \Phi(t_{\text{dec}}, \mathbf{x}_{\text{ls}}) - \Phi(t_0, 0) + 2 \int \dot{\Phi} dt + \mathbf{v}_{\text{obs}} \cdot \hat{n}
\]
(49)

where the integral is from \((t_{\text{dec}}, \mathbf{x}_{\text{ls}})\) to \((t_0, 0)\) along the path of the photon (a null geodesic). The origin \(0\) is located where the observer is. The last term, \(\mathbf{v}_{\text{obs}} \cdot \hat{n}\), is the Doppler effect from observer motion (assumed nonrelativistic), \(\mathbf{v}_{\text{obs}}\) being the observer velocity and \(\hat{n}\) the direction we are looking at. The \(\mathbf{x}_{\text{ls}}\) in \(\mathbf{x}_{\text{ls}}\) is just to remind us that \(\mathbf{x}\) lies somewhere on the last scattering sphere. In the matter-dominated universe the Newtonian potential remains constant in time, \(\dot{\Phi} = 0\), so we get a contribution from the integral only from epochs when radiation or dark energy contributions to the total energy density, or the effect of curvature, cannot be ignored.

We can understand the above result as follows. If the potential is constant in time, the blueshift the photon acquires when falling into a potential well is canceled by the redshift from climbing up the well. Thus the net redshift/blueshift caused by gravitational potential perturbations is just the difference between the values of \(\Phi\) at the beginning and in the end. However, if the potential is changing while the photon is traversing the well, this cancelation is not exact, and we get the integral term to account for this effect.

The value of the potential perturbation at the observing site, \(\Phi(t_0, 0)\), is the same for photons coming from all directions. Thus it does not contribute to the observed anisotropy. It just produces an overall shift in the observed average temperature. This is included in the observed value \(T_0 = 2.725\) K, and there is no way for us to separate it from the “correct” unperturbed value. Thus we can ignore it. The observer motion \(\mathbf{v}_{\text{obs}}\) causes a dipole (\(\ell = 1\)) pattern in the CMB anisotropy, and likewise, there is no way for us to separate from it the cosmological dipole on the last scattering sphere. Therefore the dipole is usually removed from the CMB map before analyzing it for cosmological purposes. Accordingly, we shall ignore this term also, and our final result is
\[
\left(\frac{\delta T}{T}\right)_\text{jour} = \Phi(t_{\text{dec}}, \mathbf{x}_{\text{ls}}) + 2 \int \dot{\Phi} dt.
\]
(50)

The other part, \((\delta T/T)_\text{intr}\), comes from the local temperature perturbation at \(t = t_{\text{dec}}\) and the Doppler effect, \(-\mathbf{v} \cdot \hat{n}\), from the local (baryon+photon) fluid motion at that time. Since
\[
\rho_\gamma = \frac{\pi^2}{15} T^4,
\]
(51)
the local temperature perturbation is directly related to the relative perturbation in the photon energy density,
\[
\left(\frac{\delta T}{T}\right)_\text{intr} = \frac{1}{4} \delta_\gamma - \mathbf{v} \cdot \hat{n}.
\]
(52)

We can now write the observed temperature anisotropy as
\[
\left(\frac{\delta T}{T}\right)_\text{obs} = \frac{1}{4} \delta_\gamma - \mathbf{v}^N \cdot \hat{n} + \Phi(t_{\text{dec}}, \mathbf{x}_{\text{ls}}) + 2 \int \dot{\Phi} dt.
\]
(53)

\(^5\)It is done in my course on Cosmological Perturbation Theory, Sec. 25.
(note that both the density perturbation $\delta_\gamma$ and the fluid velocity $v$ are gauge dependent).

To make further progress we now

1. consider adiabatic primordial perturbations only (like we did in Chapter 8), and

2. make the (crude) approximation that the universe is already matter dominated at $t = t_{\text{dec}}$.

For adiabatic perturbations

$$\delta_b = \delta_c = \delta_m = \frac{3}{4} \delta_\gamma. \quad (54)$$

The perturbations stay adiabatic only at superhorizon scales. Once the perturbation has entered horizon, different physics can begin to act on different matter components, so that the adiabatic relation between their density perturbations is broken. In particular, the baryon+photon perturbation is affected by photon pressure, which will damp their growth and cause them to oscillate, whereas the CDM perturbation is unaffected and keeps growing. Since the baryon and photon components see the same pressure they still evolve together and maintain their adiabatic relation until photon decoupling. Thus, after horizon entry, but before decoupling,

$$\delta_c \neq \delta_b = \frac{3}{4} \delta_\gamma. \quad (55)$$

At decoupling, the equality holds for scales larger than the photon mean free path at $t_{\text{dec}}$.

After decoupling, this connection between the photons and baryons is broken, and the baryon density perturbation begins to approach the CDM density perturbation,

$$\delta_c \leftarrow \delta_b \neq \frac{3}{4} \delta_\gamma. \quad (56)$$

We shall return to these issues as we discuss the shorter scales in Sections 9.7 and 9.8. But let us first discuss the scales which are still superhorizon at $t_{\text{dec}}$, so that Eq. (54) still applies.

### 9.6 Large scales: Sachs–Wolfe part of the spectrum

Consider now the scales $k \ll k_{\text{dec}}$, or $\ell \ll \ell_H$, which are still superhorizon at decoupling. We can now use the adiabatic condition (54), so that

$$\frac{1}{4} \delta_\gamma = \frac{1}{3} \delta_m \approx \frac{1}{3} \delta, \quad (57)$$

where the latter (approximate) equality comes from taking the universe to be matter dominated at $t_{\text{dec}}$, so that we can identify $\delta \approx \delta_m$. For these scales the Doppler effect from fluid motion is subdominant, and we can ignore it (the fluid is set into motion by gradients in the pressure and gravitational potential, but the timescale of getting into motion is longer than the Hubble time for superhorizon scale gradients).

Thus Eq. (53) becomes

$$\left(\frac{\delta T}{T}\right)_{\text{obs}} = \frac{1}{3} \delta^N + \Phi(t_{\text{dec}}, x_{\text{ls}}) + 2 \int \Phi dt. \quad (58)$$

The Newtonian relation

$$\delta = \frac{1}{4\pi G \rho a^2} \nabla^2 \Phi = \frac{2}{3} \left(\frac{1}{a H}\right)^2 \nabla^2 \Phi$$

(here $\nabla$ is with respect to the comoving coordinates, hence the $a^{-2}$) or

$$\delta_k = -\frac{2}{3} \left(\frac{k}{H}\right)^2 \Phi_k$$
does not hold at superhorizon scales (where $\delta$ is gauge dependent). A GR calculation using the Newtonian gauge gives the result:

$$
\delta_N^k = - \left[ 2 + \frac{2}{3} \left( \frac{k}{H} \right)^2 \right] \Phi_k
$$

for perturbations in a matter-dominated universe. Thus for superhorizon scales we can approximate

$$
\delta_N \approx -2\Phi
$$

and Eq. (58) becomes

$$
\left( \frac{\delta T}{T} \right)_{\text{obs}} = -\frac{2}{3} \Phi(t_{\text{dec}}, x_{ls}) + \Phi(t_{\text{dec}}, x_{ls}) + 2 \int \dot{\Phi} \, dt
$$

This explains the “mysterious” factor $1/3$ in this relation between the potential $\Phi$ and the temperature perturbation.

This result is called the *Sachs–Wolfe effect*. The first part, $\frac{1}{3} \Phi(t_{\text{dec}}, x_{ls})$, is called the *ordinary Sachs–Wolfe effect*, and the second part, $2 \int \dot{\Phi} \, dt$, the *integrated Sachs-Wolfe effect* (ISW), since it involves integrating along the line of sight. Note that the approximation of matter domination at $t = t_{\text{dec}}$, making $\dot{\Phi} = 0$, does not eliminate the ISW, since it only applies to the “early part” of the integral. At times closer to $t_0$, dark energy becomes important, causing $\Phi$ to evolve again. This ISW caused by dark energy (or curvature of the background universe, if $k \neq 0$) is called the *late Sachs–Wolfe effect* (LSW) and it shows up as a rise in the smallest $\ell$ of the angular power spectrum $C_\ell$. Correspondingly, the contribution to the ISW from the evolution of $\Phi$ near $t_{\text{dec}}$ due to the radiation contribution to the expansion law (which we ignored in our approximation) is called the *early Sachs–Wolfe effect* (ESW). The ESW shows up as a rise in $C_\ell$ for larger $\ell$, near $\ell_H$.

We shall now forget for a while the ISW, which for $\ell \ll \ell_H$ is expected to be smaller than the ordinary Sachs–Wolfe effect.

### 9.6.1 Angular power spectrum from the ordinary Sachs–Wolfe effect

We now calculate the contribution from the ordinary Sachs–Wolfe effect,

$$
\left( \frac{\delta T}{T} \right)_{\text{SW}}^{\text{obs}} = \frac{1}{3} \Phi(t_{\text{dec}}, x_{ls}),
$$

to the angular power spectrum $C_\ell$. This is the dominant effect for $\ell \ll \ell_H$.

Since $\Phi$ is evaluated at the last scattering sphere, we have, from Eq. (30),

$$
a_{\ell m} = 4\pi \ell \sum_k \frac{1}{3} \Phi_k j_\ell(kx) Y^*_{\ell m}(\hat{k}),
$$

In the matter-dominated epoch,

$$
\Phi = -\frac{3}{5} \mathcal{R},
$$

so that

$$
a_{\ell m} = -\frac{4\pi}{5} \ell \sum_k \mathcal{R}_k j_\ell(kx) Y^*_{\ell m}(\hat{k}).
$$

---

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The coefficient $a_{\ell m}$ is thus a linear combination of the independent random variables $R_k$, i.e., it is of the form

$$\sum_k b_k R_k,$$

(66)

For any such linear combination, the expectation value of its absolute value squared is

$$\left\langle \left| \sum_k b_k R_k \right|^2 \right\rangle = \sum_k \sum_{k'} b_k b_{k'}^* \langle R_k R_{k'}^* \rangle = \left( \frac{2\pi}{L} \right)^3 \sum_k \frac{1}{4\pi k^3} P_R(k) |b_k|^2,$$

(67)

where we used

$$\langle R_k R_{k'}^* \rangle = \delta_{k k'} \left( \frac{2\pi}{L} \right)^3 \frac{1}{4\pi k^3} P_R(k)$$

(68)

(the independence of the random variables $R_k$ and the definition of the power spectrum $P(k)$).

Thus

$$C_\ell = \frac{1}{2\ell + 1} \sum_m \langle |a_{\ell m}|^2 \rangle = \frac{16\pi^2}{25} \left( \frac{2\pi}{L} \right)^3 \sum_k \frac{1}{4\pi k^3} P_R(k) j_\ell(kx)^2 |Y_{\ell m}^*(k)|^2$$

$$= \frac{1}{25} \left( \frac{2\pi}{L} \right)^3 \sum_k \frac{1}{k^3} P_R(k) j_\ell(kx)^2.$$

(69)

(Although all $\langle |a_{\ell m}|^2 \rangle$ are equal for the same $\ell$, we used the sum over $m$, so that we could use Eq. (9).) Replacing the sum with an integral, we get

$$C_\ell = \frac{1}{25} \int \frac{d^3 k}{k^3} P_R(k) j_\ell(kx)^2$$

$$= \frac{4\pi}{25} \int_0^\infty \frac{dk}{k} P_R(k) j_\ell(kx)^2,$$

(70)

the final result for an arbitrary primordial power spectrum $P_R(k)$.

The integral can be done for a power-law power spectrum, $P_R(k) = A_s^2 k^{n-1}$. In particular, for a scale-invariant ($n = 1$) primordial power spectrum,

$$P_R(k) = \text{const.} = A_s^2,$$

(71)

we have

$$C_\ell = A_s^2 \frac{4\pi}{25} \int_0^\infty \frac{dk}{k} j_\ell(kx)^2 = \frac{A_s^2}{25} \frac{2\pi}{\ell(\ell + 1)},$$

(72)

since

$$\int_0^\infty \frac{dk}{k} j_\ell(kx)^2 = \frac{1}{2\ell(\ell + 1)}.$$

(73)

We can write this as

$$\frac{\ell(\ell + 1)}{2\pi} C_\ell = \frac{A_s^2}{25} = \text{const. (independent of } \ell \text{)}$$

(74)
This is the reason why the angular power spectrum is customarily plotted as $\ell(\ell + 1)C_\ell/2\pi$; it makes the ordinary Sachs–Wolfe part of the $C_\ell$ flat for a scale-invariant primordial power spectrum $P_R(k)$.

Present data is consistent with an almost scale-invariant primordial power spectrum (actually it favors a small red tilt, $n < 1$). The constant $A_s$ can be determined from the ordinary Sachs–Wolfe part of the observed $\hat{C}_\ell$. From Fig. 11 we see that at low $\ell$

$$\frac{\ell(\ell + 1)}{2\pi} \hat{C}_\ell \sim \frac{800 \mu K^2}{(2.725 K)^2} \sim 10^{-10}$$

on the average. This gives the amplitude of the primordial power spectrum as

$$P_R(k) = A_s^2 \sim 25 \times 10^{-10} = (5 \times 10^{-5})^2.$$  

We already used this result in Chapter 8 as a constraint on the energy scale of inflation.

**Exercise:** Find the $C_\ell$ of the ordinary Sachs-Wolfe effect due to a power-law power spectrum $P_R(k) = A_s^2 k^{n-1}$. Help:

$$\int_0^\infty dx x^{n-2} j_\ell^2(x) = 2^{n-4} \pi \frac{\Gamma(\ell + \frac{n}{2} - \frac{1}{2})\Gamma(3 - n)}{\Gamma(\ell + \frac{n}{2} - \frac{3}{2})\Gamma(2 - \frac{n}{2})^2}.$$ 

Take $A_s = 4.62 \times 10^{-5}$ and $n = 0.968$ (Planck 2015 central values). Give the numerical values for $C_2$ and $C_{20}$. 


9.7 Acoustic oscillations

Consider now the scales \( k \gg k_{\text{dec}} \), or \( \ell \gg \ell_H \), which are subhorizon at decoupling. The observed temperature anisotropy is, from Eq. (53)

\[
\left( \frac{\delta T}{T} \right)_{\text{obs}} = \frac{1}{4} \delta_\gamma(t_{\text{dec}}, x_{\text{ls}}) + \Phi(t_{\text{dec}}, x_{\text{ls}}) - \mathbf{v}_\gamma \cdot \hat{n}(t_{\text{dec}}, x_{\text{ls}}) + 2 \int \dot{\Phi} dt .
\]

(78)

Since we are considering subhorizon scales, we dropped the reference to the Newtonian gauge. We shall concentrate on the three first terms, which correspond to the situation at the point \((t_{\text{dec}}, x_{\text{ls}})\) we are looking at on the last scattering sphere.

Before decoupling photons are coupled to baryons. Perturbations in the baryon-photon fluid are oscillating, whereas CDM perturbations grow (slowly during the radiation-dominated epoch, and then faster during the matter-dominated epoch). Therefore CDM perturbations begin to dominate the total density perturbation \( \delta \rho \) and thus also \( \Phi \) already before the universe becomes matter dominated, and CDM begins to dominate the background energy density. Thus we make the approximation that \( \Phi \) is given by the CDM perturbation. The baryon-photon fluid oscillates in these potential wells caused by the CDM. The potential \( \Phi \) evolves at first but then becomes constant as the universe becomes matter dominated.

We shall not attempt an exact calculation of the \( \delta_\gamma \) oscillations in the expanding universe. One reason is that \( \rho_{b\gamma} \) is a relativistic fluid, and we have derived the perturbation equations for a nonrelativistic fluid only. From Sec. 8.2.7 we have that the nonrelativistic perturbation equation for a fluid component \( i \) is

\[
\ddot{\delta}_{ki} + 2H \dot{\delta}_{ki} = -\frac{k^2}{a^2} \left( \frac{\delta p_{ki}}{\bar{\rho}_i} + \Phi_k \right).
\]

(79)

The generalization of the (subhorizon) perturbation equations to the case of a relativistic fluid is considerably easier if we ignore the expansion of the universe. Then Eq. (79) becomes

\[
\ddot{\delta}_{ki} + k^2 \left( \frac{\delta p_{ki}}{\bar{\rho}_i} + \Phi_k \right) = 0 .
\]

(80)

According to GR, the density of “passive gravitational mass” is \( \rho + p = (1 + w)\rho \), not just \( \rho \) as in Newtonian gravity. Therefore the force on a fluid element of the fluid component \( i \) is proportional to \((\rho_i + p_i)\nabla \Phi = (1 + w_i)\rho_i \nabla \Phi\) instead of just \( \rho_i \nabla \Phi \), and Eq. (80) generalizes to the case of a relativistic fluid as

\[
\ddot{\delta}_{ki} + k^2 \left[ \frac{\delta p_{ki}}{\bar{\rho}_i} + (1 + w_i)\Phi_k \right] = 0 .
\]

(81)

In the present application the fluid component \( \rho_i \) is the baryon-photon fluid \( \rho_{b\gamma} \), and the gravitational potential \( \Phi \) is caused by the CDM. Before decoupling, the adiabatic relation \( \delta_b = \frac{3}{4} \delta_\gamma \) still holds between photons and baryons, and we have the adiabatic relation between pressure and density perturbations,

\[
\delta p_{b\gamma} = c_s^2 \delta \rho_{b\gamma}
\]

(82)

where

\[
c_s^2 = \frac{\delta p_{b\gamma}}{\delta \rho_{b\gamma}} \approx \frac{\delta p_{b\gamma}}{\delta \rho_{b\gamma}} = \frac{1}{3} \frac{\delta \rho_b}{\delta \rho_{b\gamma}} + \frac{1}{3} \frac{\rho_b \delta \gamma}{\rho_{b\gamma}} = \frac{1}{3 \left[ 1 + \frac{3}{4} \frac{\rho_b}{\rho_{b\gamma}} \right]} = \frac{1}{3 \left[ 1 + R \right]}
\]

(83)

\footnote{Actually the derivation is more complicated, since also the density of “inertial mass” is \( \rho_i + p_i \) and the energy continuity equation is modified by a work-done-by-pressure term. The more detailed derivation of Eq. (81) was given in Sec. 8.2.8.}
gives the speed of sound $c_s$ of the baryon-photon fluid. We defined

$$ R \equiv \frac{3}{4} \frac{\bar{\rho}_b}{\bar{\rho}_\gamma}. \quad (84) $$

We can now write the perturbation equation (81) for the baryon-photon fluid as

$$ \ddot{\delta}_{b\gamma k} + k^2 \left[ c_s^2 \delta_{b\gamma k} + (1 + w_{b\gamma}) \Phi_k \right] = 0. \quad (85) $$

The equation-of-state parameter for the baryon-photon fluid is

$$ w_{b\gamma} \equiv \frac{\bar{p}_{b\gamma}}{\bar{\rho}_{b\gamma}} = \frac{1}{3} \frac{\bar{\rho}_\gamma + \bar{\rho}_b}{\bar{\rho}_\gamma + \bar{\rho}_b} = 1 + \frac{1}{3} \frac{1 + \frac{4}{3} R}{1 + \frac{4}{3} R}, \quad (86) $$

so that

$$ 1 + w_{b\gamma} = \frac{4}{3} \frac{1 + R}{1 + \frac{4}{3} R} \quad (87) $$

and we can write Eq. (85) as

$$ \ddot{\delta}_{b\gamma k} + k^2 \left[ \frac{1}{3} \frac{1}{1 + R} \delta_{b\gamma k} + \frac{4}{3} \frac{1 + R}{1 + \frac{4}{3} R} \Phi_k \right] = 0. \quad (88) $$

For the CMB anisotropy we are interested in\(^8\)

$$ \Theta_0 \equiv \frac{1}{4} \delta_\gamma, \quad (89) $$

which gives the local temperature perturbation, not in $\delta_{b\gamma}$. These two are related by

$$ \delta_{b\gamma} = \frac{\delta \rho_{b\gamma}}{\rho_{b\gamma}} = \frac{\delta \rho_\gamma + \delta \rho_b}{\rho_\gamma + \rho_b} = \frac{\bar{\rho}_\gamma \delta_\gamma + \bar{\rho}_b \delta_b}{\rho_\gamma + \rho_b} = \frac{1 + R}{1 + \frac{4}{3} R} \delta_\gamma. \quad (90) $$

Thus we can write Eq. (85) as

$$ \ddot{\delta}_\gamma k + k^2 \left[ \frac{1}{3 + R} \delta_\gamma k + \frac{4}{3} \Phi_k \right] = 0, \quad (91) $$

or

$$ \ddot{\Theta}_{0k} + k^2 \left[ \frac{1}{3 + R} \Theta_{0k} + \frac{1}{3} \Phi_k \right] = 0, \quad (92) $$

or

$$ \Theta_{0k} + c_s^2 k^2 [\Theta_{0k} + (1 + R) \Phi_k] = 0. \quad (93) $$

If we now take $R$ and $\Phi_k$ to be constant, this is the harmonic oscillator equation for the quantity $\Theta_{0k} + (1 + R) \Phi_k$ with the general solution

$$ \Theta_{0k} + (1 + R) \Phi_k = A_k \cos c_s k t + B_k \sin c_s k t, \quad (94) $$

or

$$ \Theta_{0k} + \Phi_k = -R \Phi_k + A_k \cos c_s k t + B_k \sin c_s k t, \quad (95) $$

or

$$ \Theta_{0k} = -(1 + R) \Phi_k + A_k \cos c_s k t + B_k \sin c_s k t. \quad (96) $$

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\(^8\) The subscript 0 refers to the monopole ($\ell = 0$) of the local photon distribution. Likewise, the dipole ($\ell = 1$) of the local photon distribution corresponds to the velocity of the photon fluid, $\Theta_1 \equiv v_\gamma / 3$.\)
We are interested in the quantity $\Theta_0 + \Phi = \frac{1}{3} \delta_\gamma + \Phi$, called the effective temperature perturbation, since this combination appears in Eq. (78). It is the local temperature perturbation minus the redshift photons suffer when climbing from the potential well of the perturbation (negative $\Phi$ for a CDM overdensity). We see that this quantity oscillates in time, and the effect of baryons (via $R$) is to shift the equilibrium point of the oscillation by $-R\Phi_k$.

In the preceding we ignored the effect of the expansion of the universe. The expansion affects the preceding in a number of ways. For example, $c_s$, $w_{b\gamma}$ and $R$ change with time. The potential $\Phi$ also evolves, especially at the earlier times when radiation dominates the expansion law. However, the qualitative result of an oscillation of $\Theta_0 + \Phi$, and the shift of its equilibrium point by baryons, remains. The time $t$ in the solution (95) gets replaced by conformal time $\eta$, and since $c_s$ changes with time, $c_s \eta$ is replaced by $r_s(t)$.

We call this quantity $r_s(t)$ the sound horizon at time $t$, since it represents the comoving distance sound has traveled by time $t$.

The relative weight of the cosine and sine solutions (i.e., the constants $A_k$ and $B_k$ in Eq. (94)) depends on the initial conditions. Since the perturbations are initially at superhorizon scales, the initial conditions are determined there, and the present discussion does not really apply. However, using the Newtonian gauge superhorizon initial conditions gives the correct qualitative result for the phase of the oscillation.

We had that for adiabatic primordial perturbations, initially $\Phi = -\frac{2}{3} \mathcal{R}$ and $\frac{1}{3} \delta_\gamma^N = -\frac{2}{3} \Phi = \frac{2}{3} \mathcal{R}$, giving us an initial condition $\Theta_0 + \Phi = \frac{1}{3} \Phi = -\frac{2}{3} \mathcal{R} = \text{const.}$. (At these early times $R \ll 1$, so we don’t write the $1 + R$.) Thus adiabatic primordial perturbations correspond essentially to the cosine solution. (There are effects at the horizon scale which affect the amplitude of the oscillations—the main effect being the decay of $\Phi$ as it enters the horizon—so we can’t use the preceding discussion to determine the amplitude, but we get the right result about the initial phase of the $\Theta_0 + \Phi$ oscillations.)

Thus we have that, qualitatively, the effective temperature behaves at subhorizon scales as

$$\Theta_{0k} + (1 + R)\Phi_k \propto \cos kr_s(t),$$

Consider a region which corresponds to a positive primordial curvature perturbation $\mathcal{R}$. It begins with an initial overdensity (of all components, photons, baryons, CDM and neutrinos), and a negative gravitational potential $\Phi$. For the scales of interest for CMB anisotropy, the potential stays negative, since the CDM begins to dominate the potential early enough and the CDM perturbations do not oscillate, they just grow. The effective temperature perturbation $\Theta_0 + \Phi$, which is the oscillating quantity, begins with a negative value. After half an oscillation period it is at its positive extreme value. This increase of $\Theta_0 + \Phi$ corresponds to an increase in $\delta_\gamma$; from its initial positive value it has grown to a larger positive value. Thus the oscillation begins by the, already initially overdense, baryon-photon fluid falling deeper into the potential well, and reaching its maximum compression after half a period. After this maximum compression the photon pressure pushes the baryon-photon fluid out from the potential well, and after a full period, the fluid reaches its maximum decompression in the potential well. Since the potential $\Phi$ has meanwhile decayed (horizon entry and the resulting potential decay always happens during the first oscillation period, since the sound horizon and the Hubble length are close to each other, as the sound speed is close to the speed of light), the decompression does not bring the $\delta_\gamma$ back to its initial value (which was overdense), but the photon-baryon fluid actually becomes underdense in the potential well (and overdense in the neighboring potential “hill”). And so the oscillation goes on until photon decoupling.
Figure 19: Acoustic oscillations. The top panel shows the time evolution of the Fourier amplitudes $\Theta_{0k}$, $\Phi_k$, and the effective temperature $\Theta_{0k} + \Phi_k$. The Fourier mode shown corresponds to the fourth acoustic peak of the $C_\ell$ spectrum. The bottom panel shows $\delta_b(x)$ for one Fourier mode as a function of position at various times (maximum compression, equilibrium level, and maximum decompression).

These are standing waves and they are called *acoustic oscillations*. See Fig. 19. Because of the potential decay at horizon entry, the amplitude of the oscillation is larger than $\Phi$, and thus also $\Theta_0$ changes sign in the oscillation.

These oscillations end at photon decoupling, when the photons are liberated. The CMB shows these standing waves as a snapshot\(^9\) at their final moment $t = t_{\text{dec}}$.

At photon decoupling we have

$$\Theta_{0k} + (1 + R)\Phi_k \propto \cos kr_s(t_{\text{dec}}).$$ \hspace{1cm} (99)

At this moment oscillations for scales $k$ which have

$$kr_s(t_{\text{dec}}) = m\pi$$ \hspace{1cm} (100)

($m = 1, 2, 3, \ldots$) are at their extreme values (maximum compression or maximum decompression). Therefore we see strong structure in the CMB anisotropy at the multipoles

$$\ell = kd_A^C(t_{\text{dec}}) = m\pi \frac{d_A^C(t_{\text{dec}})}{r_s(t_{\text{dec}})} \equiv m\ell_A$$ \hspace{1cm} (101)

corresponding to these scales. Here

$$\ell_A \equiv \frac{\pi}{r_s(t_{\text{dec}})} = \frac{\pi}{\vartheta_s}$$ \hspace{1cm} (102)

is the *acoustic scale* in multipole space and

$$\vartheta_s \equiv \frac{r_s(t_{\text{dec}})}{\dot{d}_A^C(t_{\text{dec}})}$$ \hspace{1cm} (103)

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\(^9\)Actually, photon decoupling takes quite a long time. Therefore this “snapshot” has a rather long “exposure time” causing it to be “blurred”. This prevents us from seeing very small scales in the CMB anisotropy.
is the sound horizon angle, i.e., the angle at which we see the sound horizon on the last scattering surface.

Because of these acoustic oscillations, the CMB angular power spectrum $C_\ell$ has a structure of acoustic peaks at subhorizon scales. The centers of these peaks are located approximately at $\ell_m \approx m\ell_A$. An exact calculation shows that they will actually lie at somewhat smaller $\ell$ due to a number of effects. The separation of neighboring peaks is closer to $\ell_A$ than the positions of the peaks are to $m\ell_A$.

These acoustic oscillations involve motion of the baryon-photon fluid. When the oscillation of one Fourier mode is at its extreme, e.g., at the maximal compression in the potential well, the fluid is momentarily at rest, but then it begins flowing out of the well until the other extreme, the maximal decompression, is reached. Therefore those Fourier modes $k$ which have the maximum effect on the CMB anisotropy via the $\frac{1}{4} \delta_\gamma(t_{\text{dec}}, x_{\text{ls}}) + \Phi(t_{\text{dec}}, x_{\text{ls}})$ term (the effective temperature effect) in Eq. (78) have the minimum effect via the $-v \cdot \hat{n}(t_{\text{dec}}, x_{\text{ls}})$ term (the Doppler effect) and vice versa. Therefore the Doppler effect also contributes a peak structure to the $C_\ell$ spectrum, but the peaks are in the locations where the effective temperature contribution has troughs.

The Doppler effect is subdominant to the effective temperature effect, and therefore the peak positions in the $C_\ell$ spectrum are determined by the effective temperature effect, according to Eq. (101). The Doppler effect just partially fills the troughs between the peaks, weakening the peak structure of $C_\ell$. See Fig. 22.

Fig. 20 shows the values of the effective temperature perturbation $\Theta_0 + \Phi$ (as well as $\Theta_0$ and $\Phi$ separately) and the magnitude of the velocity perturbation ($\Theta_1 \sim v/3$) at $t_{\text{dec}}$ as a function of the scale $k$. This is a result of a numerical calculation which includes the effect of the expansion of the universe, but not diffusion damping (Sec. 9.8).

### 9.8 Diffusion damping

For small enough scales the effect of photon diffusion and the finite thickness of the last scattering surface ($\sim$ the photon mean free path just before last scattering) smooth out the photon distribution and the CMB anisotropy.

This effect can be characterized by the damping scale $k_D^{-1} \sim$ photon diffusion length $\sim$ geometric mean of the Hubble time and photon mean free path $\lambda_\gamma$. Actually $\lambda_\gamma$ is increasing rapidly during recombination, so a calculation of the diffusion scale involves an integral over time which includes this effect.

A calculation, that we shall not do here,\(^{10}\) gives that photon density and velocity perturbations at scale $k$ are damped at $t_{\text{dec}}$ by

$$e^{-k^2/k_D^2},$$

where the diffusion scale is

$$k_D^{-1} \sim \frac{1}{\text{few } a} \sqrt{\frac{\lambda_\gamma(t_{\text{dec}})}{H_{\text{dec}}}}.$$  \hspace{1cm} (105)

Accordingly, the $C_\ell$ spectrum is also damped as

$$e^{-\ell^2/\ell_D^2},$$

where

$$\ell_D \sim k_D d_A^2(t_{\text{dec}}).$$  \hspace{1cm} (107)

For typical values of cosmological parameters $\ell_D \sim 1500$. See Fig. 21 for a result of a numerical calculation with and without diffusion damping.

\(^{10}\)See, e.g., Dodelson [7], Chapter 8.
Figure 20: Values of oscillating quantities (normalized to an initial value $R_k = 1$) at the time of decoupling as a function of the scale $k$, for three different values of $\omega_m$, and for $\omega_b = 0.01$. $\Theta_1$ represents the velocity perturbation. The effect of diffusion damping is neglected. Figure and calculation by R. Kesktalo.
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Of the cosmological parameters, the damping scale is the most strongly dependent on $\omega_b$, since increasing the baryon density shortens the photon mean free path before decoupling. Thus for larger $\omega_b$ the damping moves to shorter scales, i.e., $\ell_D$ becomes higher (there is less damping).

(Of course, decoupling only happens as the photon mean free path becomes comparable to the Hubble length, so one might think that $\lambda_\gamma$ at $t_{\text{dec}}$ should be independent of $\omega_b$. However there is a distinction here between whether a photon will not scatter again after a particular scattering and what was the mean free path between the second-to-last and the last scattering. And $k_D$ depends on an integral over the past history of the photon mean free path, not just the last one. The factor $1/\text{few}$ in Eq. (105) comes from that integration, and actually depends on $\omega_b$. For small $\omega_b$ the $\lambda_\gamma$ has already become quite large through the slow dilution of the baryon density by the expansion of the universe, and relies less on the fast reduction of free electron density due to recombination. Thus the time evolution of $\lambda_\gamma$ before decoupling is different for different $\omega_b$ and we get a different diffusion scale.)

9.9 The complete $C_\ell$ spectrum

As we have discussed the CMB anisotropy has three contributions (see Eq. 78), the effective temperature effect,

$$\frac{1}{4} \delta_\gamma(t_{\text{dec}}, x_{ls}) + \Phi(t_{\text{dec}}, x_{ls}),$$

(108)

the Doppler effect,

$$- v \cdot \hat{n}(t_{\text{dec}}, x_{ls}),$$

(109)

and the integrated Sachs–Wolfe effect,

$$2 \int_{t_{\text{dec}}}^{t_0} \Phi(t, x(t)) dt.$$

(110)

Since the $C_\ell$ is a quadratic quantity, it also includes cross terms between these three effects.
Figure 22: The full $C_\ell$ spectrum calculated for the cosmological model $\Omega_0 = 1$, $\Omega_\Lambda = 0$, $\omega_m = 0.2$, $\omega_b = 0.03$, $A_s = 1$, $n_s = 1$, and the different contributions to it. (The calculation involves some approximations which allow the description of $C_\ell$ as just a sum of these contributions and is not as accurate as a CMBFAST or CAMB calculation.) Here $\Theta_1$ denotes the Doppler effect. Figure and calculation by R. Keskitalo.
The calculation of the full $C_\ell$ proceeds much as the calculation of just the ordinary Sachs–Wolfe part (which the effective temperature effect becomes at superhorizon scales) in Sec. 9.6.1, but now with the full $\delta T/T$. Since all perturbations are proportional to the primordial perturbations, the $C_\ell$ spectrum is proportional to the primordial perturbation spectrum $P_R(k)$ (with integrals over the spherical Bessel functions $j_\ell(kx)$, like in Eq. (70), to get from $k$ to $\ell$).

The difference is that instead of the constant proportionality factor $(\delta T/T)_{SW} = -(1/5)\mathcal{R}$, we have a $k$-dependent proportionality resulting from the evolution (including, e.g., the acoustic oscillations) of the perturbations.

In Fig. 22 we show the full $C_\ell$ spectrum and the different contributions to it.

Because the Doppler effect and the effective temperature effect are almost completely off-phase, their cross term gives a negligible contribution.

Since the ISW effect is relatively weak, it contributes more via its cross terms with the Doppler effect and effective temperature than directly. The cosmological model used for Fig. 22 has $\Omega_\Lambda = 0$, so there is no late ISW effect (which would contribute at the very lowest $\ell$), and the ISW effect shown is the early ISW effect due to radiation contribution to the expansion law. This effect contributes mainly to the first peak and to the left of it, explaining why the first peak is so much higher than the other peaks. It also shifts the first peak position slightly to the left and changes its shape.
References


