A More about General Relativity

A.1 Vectors, tensors, and the volume element

The metric of spacetime can always be written as

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \equiv \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu. \] (1)

We introduce Einstein’s summation rule: there is a sum over repeated indices (that is, we don’t bother to write down the summation sign \(\sum\) in this case). Greek (spacetime) indices go over the values 0–3, Latin (space) indices over the values 1–3, i.e., \(g_{ij}dx^i dx^j \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}dx^i dx^j\). The objects \(g_{\mu\nu}\) are the components of the metric tensor. They have, in principle, the dimension of distance squared. In practice one often assigns the dimension of distance (or time) to some coordinates, and then the corresponding components of the metric tensor are dimensionless. These coordinate distances are then converted to proper (“real” or “physical”) distances with the metric tensor. The components of the metric tensor form a symmetric 4 \(\times\) 4 matrix.

Example 1. The metric tensor for a 2-sphere (discussed in Chapter 2 as an example of a curved 2D space) has the components

\[ [g_{ij}] = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \sin^2 \vartheta \end{bmatrix}. \] (2)

Example 2. The metric tensor for Minkowski space has the components

\[ g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \] (3)

in Cartesian coordinates, and

\[ g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \vartheta \end{bmatrix} \] (4)

in spherical coordinates.

Example 3. The Robertson-Walker metric, which we discuss in Chapter 3, has components

\[ g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(r^2 - 1) & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2 \sin^2 \vartheta \end{bmatrix}. \] (5)

Note that the metric tensor components in the above examples always formed a diagonal matrix. This is the case when the coordinate system is orthogonal.

The vectors which occur naturally in relativity are four-vectors, with four components, e.g., the four-velocity. The values of the components depend on the basis \(\{e_\alpha\}\) used. Note that the index of the basis vector does not refer to a component, but specifies which one of the four basis vectors is in question. The components of the basis vectors in the basis they define are, of course,

\[ (e_\alpha)_\beta = \delta^\beta_\alpha, \] (6)
where \( \delta^\beta_\alpha \) is the Kronecker symbol, 1 if \( \alpha = \beta \), 0 otherwise.

Given a coordinate system, we have two bases (also called frames) naturally associated with it, the coordinate basis and the corresponding normalized basis. If the coordinate system is orthogonal, the latter is an orthonormal basis. When we use the coordinates to define the components of a vector, like the 4-velocity in Chapter 2, the components naturally come out in the coordinate basis. The basis vectors of a coordinate basis are parallel to coordinate lines, and their length represents the distance from changing the value of the coordinate by one unit. For example, if we move along the coordinate \( x^1 \) so that it changes by \( dx^1 \), the distance traveled is \( ds = \sqrt{g_{11} dx^1 dx^1} = \sqrt{g_{11}} dx^1 \). The length of the basis vector \( e_1 \) is thus \( \sqrt{g_{11}} \). Since in the coordinate basis the basis vectors usually are not unit vectors, the numerical values of the components give the wrong impression of the magnitude of the vector. Therefore we may want to convert them to the normalized basis

\[
e_\alpha \equiv \left( \frac{1}{\sqrt{|g_{\alpha\alpha}|}} \right) e_\alpha. \tag{7}
\]

(It is customary to denote the normalized basis with a hat over the index, when both bases are used. In the above equation there is no sum over the index \( \alpha \), since it appears only once on the left.) For a four-vector \( w \) we have

\[
w = w^\alpha e_\alpha = w^\dot{\alpha} e_\dot{\alpha}, \tag{8}
\]

where

\[
w^{\dot{\alpha}} \equiv \sqrt{|g_{\alpha\alpha}|} w^{\alpha}. \tag{9}
\]

For example, the components of the coordinate velocity of a massive body, \( v^i = dx^i / dt \) could be greater than one; the “physical velocity”, i.e., the velocity measured by an observer who is at rest in the comoving coordinate system, is \(^1\)

\[
v^i = \sqrt{g_{ii}} dt / \sqrt{|g_{00}|} dx^0, \tag{10}
\]

with components always smaller than one.

The volume of a region of space (given by some range in the spatial coordinates \( x^1, x^2, x^3 \)) is given by

\[
V = \int_V dV = \int_V \sqrt{\det [g_{ij}]} dx^1 dx^2 dx^3, \tag{11}
\]

where \( dV \equiv \sqrt{\det [g_{ij}]} dx^1 dx^2 dx^3 \) is the volume element. Here \( \det [g_{ij}] \) is the determinant of the \( 3 \times 3 \) submatrix of the metric tensor components corresponding to the spatial coordinates. For an orthogonal coordinate system, the volume element is

\[
dV = \sqrt{g_{11}} dx^1 \sqrt{g_{22}} dx^2 \sqrt{g_{33}} dx^3. \tag{12}
\]

The metric tensor is used for taking scalar (dot) products of four-vectors,

\[
w \cdot u \equiv g_{\alpha\beta} u^\alpha u^\beta. \tag{13}
\]

The (squared) norm of a four-vector \( w \) is

\[
w \cdot w \equiv g_{\alpha\beta} w^\alpha w^\beta. \tag{14}
\]

**Exercise:** Show that the norm of the four-velocity is always \(-1\).

\(^1\)When \( g_{00} = -1 \), this simplifies to \( \sqrt{g_{ii}} dx^i / dt \).
For an orthonormal basis we have

\[
\begin{align*}
e_0 \cdot e_0 &= -1 \\
e_0 \cdot e_j &= 0 \\
e_i \cdot e_j &= \delta_{ij}.
\end{align*}
\]

(15)

We shall use the short-hand notation

\[
e_\alpha \cdot e_\beta = \eta_{\alpha\beta},
\]

(16)

where the symbol \(\eta_{\alpha\beta}\) is like the Kronecker symbol \(\delta_{\alpha\beta}\), except that \(\eta_{00} = -1\).

### A.2 Contravariant and covariant components

We sometimes write the index as a subscript, sometimes as a superscript. This has a precise meaning in relativity. The component \(u^\alpha\) of a four-vector is called a contravariant component. We define the corresponding covariant component as

\[
w_\alpha \equiv g_{\alpha\beta} w^\beta.
\]

(17)

The norm is now simply

\[
w \cdot w = w_\alpha w^\alpha.
\]

(18)

In particular, for the 4-velocity we always have

\[
u_\mu v^\mu = g_{\mu\nu} u^\mu u^\nu = \frac{\,ds^2}{\,d\tau^2} = -1.
\]

(19)

We defined the metric tensor through its covariant component \(s\) (Eq. 1). We now define the corresponding covariant components \(g_{\alpha\beta}\) as the inverse matrix of the matrix \(\begin{bmatrix} g_{\alpha\beta}\end{bmatrix}\),

\[
g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma.
\]

(20)

Now

\[
g^{\alpha\beta} w_\beta = g^{\alpha\beta} g_{\beta\gamma} w^\gamma = \delta_\alpha^\gamma w^\gamma = w^\alpha.
\]

(21)

The metric tensor can be used to lower and raise indices. For tensors,

\[
\begin{align*}
A_\alpha^\beta &= g_{\alpha\gamma} A^{\gamma\beta} \\
A_\alpha^\beta &= g_{\alpha\gamma} g^{\beta\delta} A^{\gamma\delta} \\
A^{\alpha\beta} &= g^{\alpha\gamma} g^{\beta\delta} A_{\gamma\delta}.
\end{align*}
\]

(22)

Note that for the mixed components \(A^\beta_\beta \neq A^\beta_\alpha\), unless the tensor is symmetric, in which case we can write \(A^\beta_\alpha\). When indices form covariant-contravariant pairs and are summed over, as in \(A_{\alpha\beta\gamma} B^{\alpha\beta\gamma}\), the resulting quantity is invariant in coordinate transformations.

For an orthonormal basis,

\[
g_{\alpha\beta} = g^{\beta\alpha} = \eta_{\alpha\beta},
\]

(23)

and the covariant and contravariant components of vectors and tensors have the same values, except that the raising or lowering of the time index 0 changes the sign. These orthonormal components are also called “physical” components, since they have the “right” magnitude.

Note that the symbols \(\delta_{\alpha\beta}\) and \(\eta_{\alpha\beta}\) are not tensors, and the location of their index carries no meaning.
A.3 Einstein equation

From the first and second partial derivatives of the metric tensor,

$$\frac{\partial g_{\mu\nu}}{\partial x^\sigma}, \quad \frac{\partial^2 g_{\mu\nu}}{(\partial x^\sigma \partial x^\tau)} ,$$

one can form various curvature tensors. These are the Riemann tensor $R^\rho_{\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu}$, and the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, where $R$ is the Ricci scalar $g^{\alpha\beta}R_{\alpha\beta}$, also called the “scalar curvature” (not to be confused with the scale factor of the Robertson–Walker metric, which is sometimes denoted $R(t)$). We shall not discuss these curvature tensors in this course. The only purpose of mentioning them here is to be able to show the general form of the Einstein equation, before we go to the much simpler specific case of the Friedmann–Robertson–Walker universe.

In Newton’s theory the source of gravity is mass, or, in the case of continuous matter, the mass density $\rho$. According to Newton, the gravitational field $g_N$ is given by the equation

$$\nabla^2 \Phi = -\nabla \cdot g_N = 4\pi G \rho .$$

(25)

Here $\Phi$ is the gravitational potential.

In Einstein’s theory, the source of spacetime curvature is the energy-momentum tensor, also called the stress-energy tensor, or, for short, the “energy tensor” $T^{\mu\nu}$. The energy tensor carries the information on energy density, momentum density, pressure, and stress. The energy tensor of frictionless continuous matter (a perfect fluid) is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} ,$$

(26)

where $\rho$ is the energy density and $p$ is the pressure in the rest frame of the fluid. In cosmology we can usually assume that the energy tensor has the perfect fluid form. $T^{00}$ is the energy density in the coordinate frame. ($T^{0i}$ gives the momentum density, which is equal to the energy flux $T^{0i}$. $T^{ij}$ gives the flux of momentum $i$-component in $j$-direction.)

We can now give the general form of the Einstein equation,

$$G^{\mu\nu} = 8\pi GT^{\mu\nu} .$$

(27)

This is the law of gravity according to Einstein. Comparing to Newton (Eq. 25) we see that the mass density $\rho$ has been replaced by $T^{\mu\nu}$, and $\nabla^2 \Phi$ has been replaced by the Einstein tensor $G^{\mu\nu}$, which is a short way of writing a complicated expression containing first and second derivatives of $g_{\mu\nu}$. Thus the gravitational potential is replaced by the 10 components of $g_{\mu\nu}$ in Einstein’s theory.

In the case of a weak gravitational field, the metric is close to the Minkowski metric, and we can write, e.g.,

$$g_{00} = -1 - 2\Phi$$

(28)

(in suitable coordinates), where $\Phi$ is small. The Einstein equation for $g_{00}$ becomes then

$$\nabla^2 \Phi = 4\pi G(\rho + 3p) .$$

(29)

Comparing this to Eq. (25) we see that the density $\rho$ has been replaced by $\rho + 3p$. For relativistic matter, where $p$ can be of the same order of magnitude than $\rho$ this is an important modification to the law of gravity. For nonrelativistic matter, where the particle velocities are $v \ll 1$, we have $p \ll \rho$, and we get Newton’s equation.

When applied to a homogeneous and isotropic universe filled with ordinary matter, the Einstein equation tells us that the universe cannot be static, it must either expand or contract.\(^2\)

\(^2\)Equation (44) leads to $\ddot{a} < 0$, which does not allow $a(t) = \text{const}$. If we momentarily had $\dot{a} = 0$, $a$ would immediately begin to decrease.
When Einstein was developing his theory, he did not believe this was happening in reality. He believed the universe was static. Therefore he modified his equation by adding an extra term,

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu}. \quad (30)$$

The constant $\Lambda$ is called the cosmological constant. Without $\Lambda$, a universe which was momentarily static, would begin to collapse under its own weight. A positive $\Lambda$ acts as repulsive gravity. In Einstein’s first model for the universe (the Einstein universe), $\Lambda$ had precisely the value needed to perfectly balance the pull of ordinary gravity. This value is so small that we would not notice its effect in small scales, e.g., in the solar system. The Einstein universe is, in fact, unstable to small perturbations.\(^3\) When Einstein heard that the Universe was expanding, he threw away the cosmological constant, calling it “the biggest blunder of my life”.\(^4\)

In more recent times the cosmological constant has made a comeback in the form of vacuum energy. Considerations in quantum field theory suggest that, due to vacuum fluctuations, the energy density of the vacuum should not be zero, but some constant $\rho_{\text{vac}}$.\(^5\) The energy tensor of the vacuum would then have the form $T_{\mu\nu} = -\rho_{\text{vac}} g_{\mu\nu}$. Thus vacuum energy has exactly the same effect as a cosmological constant with the value

$$\Lambda = 8\pi G \rho_{\text{vac}}. \quad (31)$$

Vacuum energy is observationally indistinguishable from a cosmological constant. This is because in physics, we can usually measure only energy differences. Only gravity responds to absolute energy density, and there a constant energy density has the same effect as the cosmological constant. In principle, however, they represent different ideas. The cosmological constant is an “addition to the left-hand side of the Einstein equation”, a modification of the law of gravity, whereas vacuum energy is an “addition to the right-hand side”, a contribution to the energy tensor, i.e., a form of energy.

### A.4 Friedmann equations

We shall now apply the Einstein equation to the homogeneous and isotropic case, which leads to Friedmann–Robertson–Walker (FRW) cosmology. The metric is now the Robertson–Walker (RW) metric,

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2 \sin^2 \vartheta \end{bmatrix}, \quad (32)$$

where $K$ is a constant related to curvature of space and $a(t)$ is a function of time related to expansion of space. Calculating the Einstein tensor from this metric gives

$$G^{00} = \frac{3}{a^2}(\ddot{a}^2 + K) \quad (33)$$

$$G^{11} = -\frac{1}{a^2}(2\ddot{a}a + \dot{a}^2 + K) = G^{22} = G^{33}. \quad (34)$$

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3. “If you sneeze, the universe will collapse.”
4. This statement does not appear in Einstein’s writings, but is reported by Gamow[2].
5. In field theory, the fundamental physical objects are fields, and particles are just quanta of the field oscillations. Vacuum means the ground state of the system, i.e., fields have those values which correspond to minimum energy. This minimum energy is usually assumed to be zero (although this is not necessary). However, in quantum field theory, the fields cannot stay at fixed values, because of quantum fluctuations. Thus even in the ground state the fields fluctuate around their zero-energy value, contributing a positive energy density. This is analogous to the zero-point energy of a harmonic oscillator in quantum mechanics.
We use here the orthonormal basis (signified by the \(\hat{\text{e}}\) over the index).

We assume the perfect fluid form for the energy tensor

\[
T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}.
\] (35)

Isotropy implies that the fluid is at rest in the RW coordinates, so that \(u^\hat{\alpha} = (1, 0, 0, 0)\) and (remember, \(g^{\hat{\alpha}\hat{\beta}} = \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)\))

\[
T^{\hat{\alpha}\hat{\nu}} = \begin{bmatrix}
\rho & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p
\end{bmatrix}.
\] (36)

Homogeneity implies that \(\rho = \rho(t), \ p = p(t)\).

The Einstein equation \(G^{\hat{\alpha}\hat{\beta}} = 8\pi GT^{\hat{\alpha}\hat{\beta}}\) becomes now

\[
\frac{3}{a^2}(\dot{a}^2 + K) = 8\pi G\rho
\] (37)

\[
-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - K\frac{a^2}{\dot{a}^2} = 8\pi Gp.
\] (38)

Let us rearrange this pair of equations to\(^6\)

\[
\left(\frac{\dot{a}}{a}\right)^2 + K\frac{a}{a^2} = \frac{8\pi G}{3}\rho
\] (43)

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p).
\] (44)

These are the Friedmann equations. (“Friedmann equation” in singular refers to Eq. 43.)

References


https://blogs.scientificamerican.com/guest-blog/einsteins-greatest-blunder/

\(\text{\footnotesize{\textsuperscript{6}}}\)Including the cosmological constant \(\Lambda\) these equations take the form

\[
\frac{3}{a^2}(\dot{a}^2 + K) - \Lambda = 8\pi G\rho
\] (39)

\[
-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - K\frac{a^2}{\dot{a}^2} + \Lambda = 8\pi Gp.
\] (40)

or, in the rearranged form,

\[
\left(\frac{\dot{a}}{a}\right)^2 + K\frac{a}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho
\] (41)

\[
\frac{\ddot{a}}{a} \frac{\Lambda}{3} = -\frac{4\pi G}{3}(\rho + 3p).
\] (42)

We shall not include \(\Lambda\) in these equations. Instead, we allow for the presence of vacuum energy \(\rho_{\text{vac}}\), which has the same effect.