1. **Thermal distributions in the non-relativistic limit.** Derive the following formulae for non-relativistic Maxwell-Boltzmann statistics \( T \ll m \) and \( T \ll m - \mu \), from the general formulae given in lectures.

\[
\begin{align*}
n & = g \left( \frac{mT}{2\pi} \right)^{3/2} e^{-\frac{m-\mu}{T}} \\
\rho & = n \left( m + \frac{3T}{2} \right) \\
p & = nT \ll \rho \\
\langle E \rangle & = m + \frac{3T}{2} \\
n - \bar{n} & = 2g \left( \frac{mT}{2\pi} \right)^{3/2} e^{-\frac{m}{T}} \sinh \frac{\mu}{T}.
\end{align*}
\]

Here \( n \) is the number density of particles, and \( \bar{n} \) the number density of corresponding antiparticles.

2. **Redshift of non-relativistic particles.** Consider a distribution of non-interacting non-relativistic particles in kinetic equilibrium in an expanding universe. Using the fact that momentum redshifts as \( p_2 = (a_1/a_2)p_1 \), show that the distribution remains in kinetic equilibrium. How are the temperature and chemical potential redshifted?

3. **Primordial Black Holes.** As we will discuss in Cosmology II, the radiation-dominated epoch is not suitable for the growth of density perturbations due to gravity. An exception to this is the QCD transition, during which the equation of state may deviate significantly from \( p = \frac{1}{3} \rho \). It has been speculated that this may lead to formation of black holes. The maximum mass for such a black hole would be the mass contained within horizon radius at the time of their formation, \( M_H \equiv \frac{4\pi}{3} d_{\text{hor}}^3 \rho \).

Find \( M_H \) in units of \( m_{\odot} \) at \( T = 100 \text{ GeV} \) (before the EW transition), \( T = 150 \text{ MeV} \) (before the QCD transition), and \( T = 1 \text{ MeV} \) (before BBN).

4. **Matter–radiation equality.** The present density of matter is \( \rho_{m0} = \Omega_m \rho_c \) and the present density of radiation is \( \rho_{r0} = \rho_{\gamma0} + \rho_{\nu0} \), where \( \rho_{\gamma0} = AT_0^4 \) is the microwave background (\( T_0 = 2.725 \text{ K} \)) and \( \rho_{\nu0} = (21/8)AT_{\nu0}^4 \) is the neutrino background (we assume neutrinos are massless). Here \( A = \pi^2/15 \), and \( T_{\nu0} = 4(11)^{1/3}T_0 \). What was the age of the universe \( t_{eq} \) when \( \rho_m = \rho_r \)? (Note that in these early times—but not today—you can ignore the curvature and vacuum terms in the Friedmann equation; we make no other assumptions about the values of \( \Omega_0 \) or \( \Omega_A \), since the answer does not depend on them.) Give numerical value (in years) for the cases \( \Omega_m = 0.05 \), \( 0.3 \), and \( 1.0 \), and \( H_0 = 70 \text{ km/s/Mpc} \). What was the temperature \( (T_{eq}) \) then?