3. CURVATURE

3.2 Connection and the Covariant Derivative

- On a curved manifold, it is not clear how to compare directions at different points.

- This is related to the question: how to generalize the partial derivative of a vector field (a tensor in Minkowski space) to a curved manifold so that it is a tensor.

- More generally, want a **covariant derivative operator** $\nabla$

  \[(r,s) \text{ tensor field} \rightarrow (r,s+1) \text{ tensor field}\]

  There are a number of properties we want for $\nabla$:

  1) **Linearity** $\nabla(T+S) = \nabla T + \nabla S$

  2) **Leibniz rule** $\nabla(T\otimes S) = \nabla T \otimes S + T \otimes \nabla S$

  3) keep the partial derivative of a scalar as is (already a tensor)

    $\nabla f = df \quad \text{i.e.} \quad (\nabla f)_x = \nabla_x f = \partial_x f = f_x$

- It can be shown that from these follows that $\nabla_a = \partial_a + \text{linear transformation}$

  e.g. $\nabla_a V^\mu = \partial_a V^\mu + \Gamma^\mu_{\alpha\beta} V^\alpha$

  $\Gamma^\mu_{\alpha\beta}$ form matrix

  The $n^2$ numbers, for each point $p \in M$, (which depend on the geometry of the manifold and the coordinate system), $\Gamma^\mu_{\alpha\beta}$ are called **connection coefficients** (do not form a tensor).

- From the condition that $V^\mu_{\ \alpha}$ transforms as (1,1) tensor, we get (exercise)

  $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha} \quad (3.1)$

  where $X_{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^\beta} x_\gamma$
Similarly \( \nabla_k \omega = \partial_k \omega + (\Gamma^g_{k\mu}) \omega = \partial_k \omega + \Gamma^g_{k\mu} \omega = \omega_{j\mu} \)

There is no reason why \( \Gamma^g_{k\mu} \) should be the same as \( \Gamma^g_{k\mu} \)

(we do, however, get the same unit transformation rule for \( \lambda \)). But once we introduce another requirement:

4) Covariant contractions \( \nabla_m (T^g_{\alpha\beta}) = (\nabla T)^g_{\alpha\beta} \mu \nu \)

(which is equivalent to \( \nabla_m \delta^g_{\alpha\beta} = 0 \), i.e., the tensor tensor is "covariantly constant")

we can easily derive that from

\[ \nabla_k (V^m \omega) = \partial_k (V^m \omega) \Rightarrow \tilde{\Gamma}^{g}_{\mu\nu} = -\Gamma^g_{\mu\nu} \]

* In the same way we get for tensors of any type:

\[ \begin{align*}
\varphi_{\alpha}^k &= \varphi_{\alpha}^k, \\
\omega_{\alpha\beta} &= \omega_{\alpha\beta} + \Gamma^g_{\alpha\beta} \omega \xi \\
\omega_{\alpha\beta} &= \omega_{\alpha\beta} + \Gamma^g_{\alpha\beta} \omega \xi \\
T_{\alpha\beta} &= T_{\alpha\beta} + \Gamma^g_{\alpha\beta} T_{\eta\delta} + \Gamma^g_{\eta\delta} T_{\alpha\beta}
\end{align*} \]  \( (3.2) \)

* Thus we have a covariant derivative on our manifold, if we are given a connection \( \Gamma^g_{\alpha\beta} \) on it. Now suppose we are given two different connections, \( \Pi \) and \( \tilde{\Pi} \), for the same manifold. From the transformation law \( (3.1) \) we immediately see that the difference

\[ S^g_{\mu\nu} = \Pi^g_{\mu\nu} - \tilde{\Pi}^g_{\mu\nu} \]

is a tensor (the XXX parts cancel)

* On the other hand, if \( \Pi_{\mu\nu} \) is a connection, and \( A_{\mu\nu} \) a (1,2) tensor, then \( \tilde{\Pi}_{\mu\nu} = \Pi_{\mu\nu} + A_{\mu\nu} \) is also an acceptable connection (since \( A_{\mu\nu} \) is constant)

* Switching the lower indices, \( \tilde{\Pi}_{\mu\nu} = \Pi_{\nu\mu} \) gives also an acceptable connection (since in Eq. 3.1 the \( X^g_{\nu\mu} = X^g_{\mu\nu} \))
The difference \( \Gamma^\alpha_{\mu
u} \equiv \Gamma^\alpha_{\mu
u} - \Gamma^\alpha_{\nu\mu} = 2\Gamma^\alpha_{[\mu\nu]} \) is called the torsion tensor (associated with the connection \( \Gamma^\alpha_{\mu
u} \)).

Given a metric \( g_{\mu\nu} \) on the manifold, we get a unique connection associated with the metric, by requiring two more conditions:

5) Connection is torsion-free: \( \Gamma^\alpha_{\mu
u} = \Gamma^\alpha_{\nu\mu} \)

6) Metric compatibility: \( \nabla_\alpha g_{\mu\nu} = 0 \) \( (g_{\mu\nu} = 0) \)

From metric compatibility follows some useful properties:

\[
\begin{align*}
\nabla_\alpha g_{\mu\nu} &= 0 \\
\nabla_\alpha g^{\mu\nu} &= 0
\end{align*}
\]

and \( \nabla_\alpha \) commutes with raising/lowering indices

\[
g_{\mu\nu} \nabla_\alpha V^\nu = \nabla_\alpha (g_{\mu\nu} V^\nu) = \nabla_\alpha V^\mu
\]

To show that such a connection (satisfying 5 & 6) a) exists and b) is unique:

Write out the condition \( \nabla_\alpha g_{\mu\nu} = 0 \) for different index permutations

\[
\begin{align*}
\nabla_\alpha g_{\mu\nu} &= \partial_\alpha g_{\mu\nu} - \Gamma^\lambda_{\mu\alpha} g_{\lambda\nu} - \Gamma^\lambda_{\nu\alpha} g_{\mu\lambda} = 0 \\
\nabla_\alpha g_{\mu\nu} &= \partial_\alpha g_{\mu\nu} - \Gamma^\lambda_{\nu\alpha} g_{\mu\lambda} - \Gamma^\lambda_{\mu\alpha} g_{\nu\lambda} = 0 \\
\nabla_\alpha g_{\mu\nu} &= \partial_\alpha g_{\mu\nu} - \Gamma^\lambda_{\mu\alpha} g_{\nu\lambda} - \Gamma^\lambda_{\nu\alpha} g_{\mu\lambda} = 0
\end{align*}
\]

\[
\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\alpha\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\alpha} = 0
\]

proving that: if a metric-compatible torsion-free connection exists, it must be this. (b)

You can check yourself, that this indeed is a connection, i.e. transforms according to (3.1). (a)

* These conditions are not part of the general definition of a connection. They are specific and
  a special one on a manifold w a metric. The connection in GR is this one.

In some alternative theories of gravity, conditions 5 & 6 are dropped, to give the
connection a more independent role, not so tightly related to the metric.
This connection, (3.3), is called the Christoffel connection (or Levi-Civita conn. or Riemann conn.) Its coefficients are often called Christoffel symbols. It is the connection of General Relativity. We shall use it from here on.

- The Christoffel connection satisfies \( \Gamma^\nu_{\mu\nu} = \frac{1}{\sqrt{\gamma}} \partial_{\nu} \sqrt{\gamma} \) (exercise)

This leads to a simple result for the (covariant) divergence of a vector:

\[
\nabla_{\mu} V^\nu = \partial_{\mu} V^\nu + \Gamma^\nu_{\mu\nu} V^\nu = \partial_{\mu} V^\nu + \frac{1}{\sqrt{\gamma}} \partial_{\nu} \sqrt{\gamma} V^\nu
\]

\[
= \frac{1}{\sqrt{\gamma}} \partial_{\nu} (\sqrt{\gamma} V^\nu)
\]

- The covariant derivative will be the tool by which SR equations will be converted to GR equations; partial derivatives will be replaced by covariant derivatives:

\[
\partial_{\mu} \rightarrow \nabla_{\mu} \quad \text{or} \quad \partial_{\alpha} \rightarrow \Gamma_{\alpha\beta}^{\gamma} (\text{this will be discussed in Chapter 4})
\]

Note that we do not always need to convert \( \partial_{\nu} \rightarrow \nabla_{\nu} \) to obtain tensors on a curved manifold. In particular, (for the Christoffel connection)

- exterior derivative

\[
(\text{d}w)_{\mu\nu} = 2\partial_{[\mu} \omega_{\nu]} = 2\nabla_{[\mu} \omega_{\nu]}
\]

- vector-field commutator

\[
[\psi, \psi']^\mu = \psi^\kappa \psi'^\nu, \kappa - \psi^\nu \psi'^\mu, \kappa = \psi^\kappa \psi'^\mu, \kappa - \psi^\nu \psi'^\nu, \kappa
\]

since the antisymmetric part of the \( \Gamma^\nu_{\alpha\beta} \) cancels out. (If the connection is not torsion-free, the latter equalities do not hold; the more fundamental definitions of the exterior derivative and the commutator are in terms of the partial derivative (they are defined even when no connection or metric is specified).
The determinant of the matrix \( g \equiv \det \left[ g_{\alpha \beta} \right] \) (\( g < 0 \) for GR) has an important role in the differential and integral calculus on manifolds.

First a reminder of a few basic results of matrix algebra \( M_3 \equiv \text{Helmholtz, Athena, Berkeley} \).

Let the elements of the matrix \( A \) be \( a_{ij} \).

**Cofactor:** \( \text{cof} (a_{ij}) = (-1)^{i+j} A_{ij} \) \( (5.16) \)

\(2\) subdeterminant of matrix with row \( i \) and column \( j \) removed

\[ \det A = \sum_j a_{ij} \text{cof} (a_{ij}) = \sum_i a_{ij} \text{cof} (a_{ij}) \] \( (5.15) \)

**Inverse matrix** \( B = [b_{ij}] = A^{-1} : b_{ij} = \frac{1}{\det A} \text{cof} (a_{ij}) \) \( (5.25) \)

Apply these now to the matrix of metric tensor components:

\[ (5.25) \Rightarrow g^{\alpha \beta} = \frac{1}{g} \text{cof} (g_{\alpha \beta}) \Rightarrow \text{cof} (g_{\alpha \beta}) = g \cdot g^{\alpha \beta} \]

\[ (5.15) \Rightarrow g = \sum_b g_{\alpha \beta} \text{cof} (g_{\alpha \beta}) \quad \text{note that cof} (g_{\alpha \beta}) \text{ is independent of } g_{\alpha \beta} \]

\[ \Rightarrow \frac{\partial g}{\partial g_{\alpha \beta}} = \text{cof} (g_{\alpha \beta}) \Rightarrow \frac{\partial g}{\partial g_{\alpha \beta}} \partial g_{\alpha \beta} = \sum_{\alpha \beta} g_{\alpha \beta} \text{cof} (g_{\alpha \beta}) \]

- Let the sum rule now be **RESTORED**! We get

\[ \sum_{\alpha \beta} g_{\alpha \beta} \text{cof} (g_{\alpha \beta}) = g \cdot g^{\alpha \beta} \left( \Gamma^\alpha_{\beta \gamma} + \Gamma^\alpha_{\gamma \beta} \right) = g \cdot (\Gamma^\alpha_{\beta \gamma} + \Gamma^\alpha_{\gamma \beta}) = 2g_{\alpha \beta} \]

\[ \Rightarrow \frac{\partial g}{\partial g_{\alpha \beta}} \partial g_{\alpha \beta} = \frac{1}{2g} \partial g = \frac{1}{2g} \ln |g| = \partial_{\alpha} \ln |g| = \frac{\partial_{\alpha} \ln |g|}{\sqrt{|g|}} \]

- What is the role of \( g \), or \( \sqrt{|g|} \)? According to a fundamental theorem of integral calculus, the volume element \( d^N x \) transforms as

\[ d^N x' = \det [X^\alpha_\beta] d^N x \]

Since \( g_{\alpha \beta} = X^\alpha_\beta X^\beta_\beta \quad \text{gcd} \Rightarrow g' = \det [g_{\alpha \beta}] = \det [X^\alpha_\beta]^2 \quad g \]

(Thus \( g \) is not a scalar. Quotients which transform like tensors except for extra factors at the Jacobian determinant \( \det [X^\alpha_\beta] \) are called tensor densities of weight \( n \). Two \( g \) is a scalar density of weight \(-2\).)

\[ \Rightarrow \sqrt{|g|} \sqrt{d^N x} = \sqrt{\det [X^\alpha_\beta]} \sqrt{|g'}| \sqrt{d^N x} \quad \text{is an invariant volume element.} \]
As promised, the connection provides us with a way of comparing vectors (or tensors) at different points of the manifold. However, this comparison turns out to depend on the path, with which we choose to connect the points.

The covariant derivative of a vector (or a tensor field) gives the "rate of change" of the vector.

Change with respect to what? With respect to a parallel-transported vector.

Before defining parallel transport on a manifold, consider it on the 2-sphere where the concept should be intuitively clear.

The result of parallel-transporting a vector from \( p \) to \( q \) depends on the path of transport.

In GR there is no unique way of comparing two vectors at different points. For example we can compare two 4-velocities at the same point to find out the relative velocity of two objects—which cannot exceed the speed of light. But the concept of a relative velocity of two objects at different points (e.g., two galaxies) does not exist.

In flat space the constancy of a tensor \( T^{\mu \nu} \) along a curve \( x^\alpha(\lambda) \) would just mean the constancy of the components:

\[
\frac{d}{d\lambda} T^{\mu \nu} = \frac{dx^\alpha}{d\lambda} \nabla_\alpha T^{\mu \nu} = 0
\]

In curved spacetime we replace the partial \( \partial_\lambda \) with the covariant derivative, defining the directional covariant derivative (map \( (v, s) \to (v, s) \) tensors)

\[
\frac{D}{d\lambda} = \frac{dx^\alpha}{d\lambda} \nabla_\alpha
\]

Parallel transport of \( T^{\mu \nu} \) along \( x^\alpha(\lambda) \) is defined by the requirement

\[
\left( \frac{D T}{d\lambda} \right)^{\mu \nu} = \frac{dx^\alpha}{d\lambda} \nabla_\alpha T^{\mu \nu} = 0,
\]

the parallel transport equation.
For a vector $V^\mu$, the parallel transport eq is

$$\left( \frac{D}{d\lambda} V^\mu \right)^M = \frac{d\lambda}{d\lambda} \left( \gamma_{\alpha\mu} V^\nu + \Gamma^\mu_{\alpha\nu} V^\nu \right) = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu} \frac{dx^\alpha}{d\lambda} V^\nu = 0$$

We can think of the parallel transport eq. as an initial value problem: given a tensor at an initial point $p$, and a path $x^\mu(\lambda)$ leaving away from it, who have a 1st order diff.eq to solve the parallel-transported tensor along this path.

From the metric compatibility follows that the metric always satisfies the parallel transport eq.

$$\frac{D}{d\lambda} g_{\mu\nu} = \frac{dx^\lambda}{d\lambda} V^\lambda g_{\mu\nu} = 0$$

$$\Rightarrow \quad U \cdot V = g_{\mu\nu} U^\mu V^\nu \quad \text{is conserved}$$

if $U$ and $V$ are parallel transported

$$\Rightarrow \quad \text{length of a vector, angle between vectors are conserved in parallel transport}$$

Geodesics: A geodesic is the generalization of a straight line to curved space.

In Euclidean space a straight line

1) is the shortest way between two points
2) parallel transports its own tangent vector (i.e. all tangent vectors are parallel to each other)

Generalize from the 2nd property: we define a geodesic as a curve along which the tangent vector is parallel transported:

$$\frac{D}{d\lambda} \frac{dx^\mu}{d\lambda} = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\nu} \frac{dx^\alpha}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \quad \Rightarrow \quad x^\mu(\lambda)$$

the geodesic equation. (This is actually a condition, not only on the set of points making up the curve, but also on the parametrisation: if the parameter does not increase uniformly along the curve, i.e., is not an affine parameter, the length of the tangent vector keeps changing, and the parametrized curve is not geodesic.)
Consider now the 1st property. In GR all light-like curves have zero length.

Consider therefore timelike geodesics. Instead of length we then speak of proper
distance.

\[ \ell_{pq} = \int_{p}^{q} \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \, d\lambda \]

Find no path from \( p \) to \( q \) which extremizes \( \ell_{pq} \) using calculus of
variations:

\[ \text{vary path } x^\mu(\lambda) \rightarrow x^\mu(\lambda) + \delta x^\mu(\lambda) \]

\[ \Rightarrow g_{\mu\nu} \rightarrow g_{\mu\nu} + g_{\mu\nu} \delta x^\nu \]

\[ \delta \ell_{pq} = \frac{1}{2} \int_{p}^{q} \left( -g_{\mu\nu,\sigma} \frac{dx^\mu}{d\lambda} \frac{dx^\sigma}{d\lambda} \right)^{1/2} \left( g_{\mu\nu,\sigma} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) d\lambda \]

\[ = \int \left( -\frac{1}{2} g_{\mu\nu,\sigma} \frac{dx^\mu}{d\lambda} \frac{dx^\sigma}{d\lambda} \delta x^\nu - g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right) d\lambda \]

Notice, for later use, that \((-\frac{1}{2})\) the integrand in the variation of \( L \equiv \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \) \( (\cdot \equiv \frac{d}{d\lambda}) \)

Integrate the latter term by parts (variation vanishes at end points)

\[ \delta x^\sigma \]

\[ = \int \left( -\frac{1}{2} g_{\mu\nu,\sigma} \dot{x}^{\mu} \dot{x}^{\nu} + g_{\mu\nu,\sigma} \dot{x}^{\mu} \dot{x}^{\nu} + g_{\mu\sigma} \dot{x}^{\mu} \right) \delta x^\nu d\lambda = 0 \quad \forall \delta x^\nu \]

\[ \frac{1}{2} \left( g_{\mu\nu,\sigma} \dot{x}^{\mu} \dot{x}^{\nu} + g_{\mu\nu,\sigma} \dot{x}^{\mu} \dot{x}^{\nu} \right) \dot{x}^{\mu} \dot{x}^{\nu} = 0 \quad \forall \delta x^\nu \]

\[ \Rightarrow g_{\mu\nu,\sigma} \dot{x}^{\mu} + \frac{1}{2} \left( g_{\nu\mu,\sigma} + g_{\mu\nu,\sigma} - g_{\mu\nu,\sigma} \right) \dot{x}^{\mu} \dot{x}^{\nu} = 0 \quad \text{Geodesic equation} \]

\[ \begin{align*}
\dot{x}^3 + \frac{1}{2} g^{\sigma\tau} \left( g_{\nu\mu,\sigma} + g_{\mu\nu,\tau} - g_{\mu\nu,\sigma} \right) \dot{x}^{\mu} \dot{x}^{\nu} &= 0 \\
\end{align*} \]

Note that this gives the Christoffel connection.

\[ \textit{\textbf{2.}} \text{We parameterize all paths so that } \lambda_0 \text{ and } \lambda_1 \text{ have fixed values.} \]
Thus the curve which extremizes proper time between two points, satisfies the geodesic equation, i.e., is a geodesic. It turns out that this corresponds to maximizing proper time. (Consider, e.g., the twin "paradox": The twin that stays home (on her geodesic) gets older than the twin that traveled around, accelerating now and then, i.e., deviating from geodesics.) Note, however, that this is a local maximum, i.e., with small variations of the path. A geodesic is not guaranteed to give the maximum time globally.

The Euler–Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^\sigma} \right) - \frac{\partial L}{\partial x^\sigma} = 0$$

for the Lagrangian $L = \frac{1}{2} g_{\mu\nu}(x^\sigma) \dot{x}^\mu \dot{x}^\nu$

provide a quicker way to calculate the geodesic eqs and/or the Christoffel symbols for a given metric $g_{\mu\nu}$ than the definition

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left( \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)$$
Example: Consider the metric \( ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2) \) of the flat expanding universe \( -dt^2 + a(t)^2 \delta_{ij} \delta_{k\ell} dx^i dx^j \).

\[
L = \frac{1}{2} g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu = -\frac{1}{2} \dot{t}^2 + \frac{1}{2} a(t)^2 \delta_{ij} \dot{x}^i \dot{x}^j = -\frac{1}{2} \dot{t}^2 + \frac{1}{2} a(t)^2 (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)
\]

\[
\frac{\partial L}{\partial \dot{t}} = -\ddot{t} \quad \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{t}} \right) = -\ddot{t} \quad \quad \frac{\partial L}{\partial \dot{t}} = a(t)^2 \delta_{ij} \dot{x}^i \dot{x}^j \quad \quad \text{(Note that)} \quad \frac{\partial L}{\partial t} = \frac{\partial}{\partial t} \text{non-s}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{t}} \right) - \frac{\partial L}{\partial \dot{t}} = -\ddot{t} - a(t)^2 \delta_{ij} \dot{x}^i \dot{x}^j = -\ddot{t} - a(t)^2 (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0 \quad \mid x = 0 \rightarrow -1
\]

\[
\Rightarrow \dot{t} + a(t)^2 (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \dot{t} + a(t)^2 \delta_{ij} \dot{x}^i \dot{x}^j = \dot{t} + a(t)^2 \delta_{ij} \dot{x}^i \dot{x}^j = 0
\]

\[
\Rightarrow \Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^0_{i0} = 0 , \quad \Gamma^0_{ij} = a(t)^2 \delta_{ij}
\]

\[
\frac{\partial L}{\partial \dot{x}} = a^2 \dot{x} \quad \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}} \right) = a^2 \ddot{x} + 2a \dot{a} \dot{t} \dot{x} \quad \frac{\partial L}{\partial \dot{x}} = 0
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial \dot{x}} = a^2 \ddot{x} + 2a \dot{a} \dot{t} \dot{x} = 0 \quad \mid x = 0 \rightarrow a^2
\]

\[
\Rightarrow \ddot{x} + 2a \frac{\dot{a}}{a} \dot{x} = \ddot{x} + \Gamma_{ij}^{\mu} \dot{x}^i \dot{x}^j = 0
\]

\[
\Rightarrow \Gamma_{00}^1 = \Gamma_{ij}^{1} = 0 \quad , \quad \Gamma_{0i}^1 = \Gamma_{i0}^1 = \frac{\dot{a}}{a} \quad , \quad \Gamma_{ij}^1 = \Gamma_{i0}^1 = 0
\]

We summarize that \( a(t)^2 \delta_{ij} \), \( \Gamma_{ij}^1 = \frac{\dot{a}}{a} \delta_{ij} \). This is not yet zero.

[Diagram or Figure]
3.4 Properties of Geodesics and Prolate Normal Coordinates

- We get from SR tensor eqs to GR tensor eqs by replacing ordinary derivatives with covariant derivatives. In SR the eq. of motion of a free particle is

$$ \alpha^\mu = U^\nu \alpha^\nu = 0 $$

$\Rightarrow$ In GR the eq. of motion of a freely falling test particle in

$$ \alpha^\mu = U^\nu \ddot{u}^\nu + \Gamma^\mu_{\nu \gamma} U^\nu \dot{u}^\gamma = \frac{d}{d\tau} \left( \frac{dx^\mu}{d\tau} \right) + \Gamma^\mu_{\nu \gamma} \frac{dx^\nu}{d\tau} \frac{dx^\gamma}{d\tau} $$

$$ = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu \gamma} \frac{dx^\nu}{d\tau} \frac{dx^\gamma}{d\tau} = 0 $$

- Freely falling test particles follow geodesics.

- If this one (non-covariant) form occurs on the particle, we can put them on the RHS, e.g. for a charged particle in an electromagnatic field

$$ \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu \gamma} \frac{dx^\nu}{d\tau} \frac{dx^\gamma}{d\tau} = \frac{q}{m} Fe^\nu \frac{dx^\nu}{d\tau} $$

- Since $\sigma^\mu = \mu u^\mu$, we can also write the geodesic eq as

$$ \sigma^\mu \sigma_\nu \sigma^\nu = 0 \quad \text{"particles keep moving in the direction their 4-momentum in pointing"} $$

This form of the geodesic eq. can be used also for lightlike geodesics, where $\tau$ cannot be used as a parameter. If a parametrised curve $x^\mu(\tau)$ is a geodesic for some parametrisation $\tau$, it is also geodesic for another parametrisation $\tilde{\tau} = a \tau + b$ ($a, b$ constants). It is often convenient to choose $\tau$ for a lightlike geodesic so that

$$ \sigma^\mu = \frac{dx^\mu}{d\tau} \quad \text{( photon momentum) } $$

- In general we can then write the geodesic eq as

$$ \frac{d\sigma^\mu}{d\tau} + \Gamma^\mu_{\nu \gamma} \sigma^\nu \sigma^\gamma \sigma^\nu = 0 \quad \text{where } \lambda = \frac{1}{m} \text{ for massive particles} $$

*) i.e. we ignore the gravitational effect of the particle itself on the geometry of the spacetime through which it moves.
Photons in an expanding universe

\[ ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2) \]

- Photons follow lightlike geodesics \( x^\mu(\lambda) \)

geodesic eqs.

\[
\begin{align*}
\ddot{t} + \alpha a' (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) &= 0 \\
\dot{x} + 2 \frac{a}{a'} \dot{t} \dot{x} &= 0 \\
\dot{y} + 2 \frac{a}{a'} \dot{t} \dot{y} &= 0 \\
\dot{z} + 2 \frac{a}{a'} \dot{t} \dot{z} &= 0
\end{align*}
\]

Consider motion in the \( x \) direction: \( \dot{y} = \dot{z} = 0 \) initially \( \Rightarrow \dot{y} = \dot{z} = 0 \)

\( \Rightarrow \dot{y} = \dot{z} = 0 \) always

Besides the geodesic eqs, photons also satisfy the lightlike condition

\[ ds^2 = -dt^2 + a(t)^2 dx^2 = 0 \quad \Rightarrow \quad dt = \pm a dx \]

Thus we have 3 eqs:

1. \( \frac{d^2 t}{d\lambda^2} + \alpha a' (\frac{dx}{d\lambda})^2 = 0 \)
2. \( \frac{d^2 x}{d\lambda^2} + 2 \frac{a}{a'} \frac{dx}{d\lambda} \frac{dt}{d\lambda} = 0 \)
3. \( \frac{dx}{d\lambda} = \pm \frac{1}{a} \frac{dt}{d\lambda} \)

Choose photon moving in positive \( x \) direction:

(3) \( \Rightarrow \frac{dx}{d\lambda} = \frac{1}{a} \frac{dt}{d\lambda} \) \( (3') \)

\( \Rightarrow \) (1) becomes \( \frac{d^2 t}{d\lambda^2} + \alpha \frac{1}{a'} (\frac{dt}{d\lambda})^2 = 0 \)

The solution is \( \frac{dt}{d\lambda} = \frac{\omega_0}{a} \), \( \omega_0 = \text{const} \) \( (4) \)

One easily sees that (3') and (4) also satisfy eq. (2).

Given the function \( a(t) \), one can solve \( t(\lambda) \) from (4)

and \( x(\lambda) \) from (3')

We obtained for the photon 4-momentum

\[ p^\mu = \frac{dx^\mu}{d\lambda} = \left( \frac{\omega_0}{a}, \frac{\omega_0}{a^2}, 0, 0 \right) \]
Photon 4-momentum $p^\mu = \left( \frac{\omega}{a}, \frac{\omega}{a^2}, 0, 0 \right)$

We should not rush to interpret $p^0, p^i$ as the photon energy and 3-momentum. In SR, vector and tensor components are observer-dependent. In GR, they are worse: coordinate-dependent.

We should either go to the observer’s locally inertial coordinates for an interpretation or calculate scalar quanmities (they are not independent).

The photon 4-momentum is indeed light-like

$$g_{\mu\nu}p^\mu p^\nu = -(\frac{\omega}{a})^2 + a^2 \left( \frac{\omega}{a^2} \right)^2 = 0$$

The energy $E$ observed by an observer with 4-velocity $u^\mu$ is

$$E = -u^\mu p_\mu = -g_{\mu\nu}u^\mu p^\nu$$ (a scalar quantity)

A comoving observer has constant space coords’ $\frac{dx^i}{dt} = 0$

$\Rightarrow \ u^\mu = \frac{dx^\mu}{dt} = (u^0, 0, 0, 0)$

Find $u^0$ from $u^0 u_0 = g_{\mu\nu}u^\mu u_\nu = -1$

For comoving observer $g_{\mu\nu}u^\mu u_\nu = g_{00}u^0 u^0 = -1 \Rightarrow u^0 = \sqrt{-1}$

Here $g_{00} = -1 \Rightarrow u^0 = 1$

$$\therefore E = -g_{\mu\nu}u^\mu p^\nu = -(-1) \cdot \frac{\omega}{a} - a^2 \cdot 0. \frac{\omega}{a^2} = \frac{\omega}{a} \hspace{1cm} (= t\omega)$$

$\Rightarrow E \propto \frac{1}{a}$  \hspace{1cm} \text{cosmological redshift}

$\nu = \frac{C}{\lambda} = \frac{2\pi C}{\omega} = \frac{2\pi}{\omega_0} \cdot a \propto a$

$\omega_0 =$ angular frequency when $a = 1$.
3.6 The Riemann Curvature Tensor

Flatness $\Rightarrow$ Curvature

On a flat manifold (Euclidean or Minkowski space):

1) Parallel transport around a closed loop leaves a vector unchanged
2) Covariant derivative of tensors commutes $V^g_{;j\nu} = V^g_{j\nu}$
3) Initially parallel geodesics remain parallel

Now consider a curved manifold

1) The change in a vector related to curvature within the whole loop.

For a local measure of curvature consider an infinitesimal loop defined by two (infinitesimal) vectors $a$ and $b$.

Transport a vector $v$ around $\Rightarrow v + \delta v$

$$\delta v \ll a, b, v \Rightarrow \delta v^g = R^g_{\phantom{g}j\nu\kappa} V^\kappa a^\mu b^\nu$$

Riemann tensor

Interchange $a \leftrightarrow b$: travel in opposite direction, $\Rightarrow -\delta v^g$

$$\Rightarrow R^g_{\phantom{g}j\nu\kappa} = - R^g_{\phantom{g}k\nu\jmath}$$

2) Now $V^g_{;j\nu} \neq V^g_{j;\nu}$ . Consider the commutator of two covariant derivatives

$$[\nabla_{\mu}, \nabla_{\nu}] V^g = \nabla_{\mu} \nabla_{\nu} V^g - \nabla_{\nu} \nabla_{\mu} V^g = V^g_{;\mu\nu} - V^g_{;j\mu}$$

$$= \partial_{\mu}(\nabla_{\nu} V^g) - \Gamma^h_{\mu\nu} \nabla_{\nu} V^g + \Gamma^h_{\mu\kappa} \nabla_{\nu} V^\kappa - (\mu \leftrightarrow \nu)$$

$$= \partial_{\mu} \nabla_{\nu} V^g + \partial_{\nu} \Gamma^h_{\mu\nu} V^g + \Gamma^h_{\mu\kappa} \partial_{\nu} V^\kappa - \Gamma^h_{\mu\nu} \partial_{\mu} V^g - \Gamma^h_{\mu\nu} \partial_{\nu} V^g$$

$$+ \Gamma^h_{\mu\kappa} \partial_{\nu} V^\kappa + \Gamma^h_{\mu\nu} \partial_{\nu} V^\mu - (\mu \leftrightarrow \nu)$$

$$= (\partial_{\mu} \Gamma^h_{\nu\kappa} - \partial_{\nu} \Gamma^h_{\mu\kappa} + \Gamma^h_{\mu\kappa} \partial_{\nu} - \Gamma^h_{\mu\nu} \Gamma^h_{\kappa\lambda} - \Gamma^h_{\mu\nu} \Gamma^h_{\kappa\lambda} - \Gamma^h_{\mu\nu} \Gamma^h_{\kappa\lambda}) V^g$$

$$\equiv R^g_{\phantom{g}j\nu\kappa} \text{ (antisymmetry in } \nu\kappa \text{ evident from definition)}$$

Exercise: (i) Show that our case 1) leads to the same

(ii) Show that $R^g_{\phantom{g}j\nu\kappa}$ transforms as a (1,3) tensor

(We leave also to study of 3) geodesic deviation as an exercise.)
The Riemann tensor is a sufficient measure of the curvature of the manifold. We earlier noted informally that

flattens \iff\text{ind. system where } g_{\mu\nu} = \text{const}

It can be shown that

\text{ind. system where } g_{\mu\nu} = \text{const} \iff R^g_{\mu\nu} = 0 \quad \text{(everywhere)}

The \iff\, is easy to show:

\[ R^g_{\mu\nu} = 0 \quad \text{everywhere} \iff \Gamma^g_{\mu\nu} = 0 \quad \text{everywhere} \iff R^g_{\mu\nu} = 0 \]

Since \( R^g_{\mu\nu} \) is a tensor, it is zero in any ind. system.

We skip the proof of \iff\, (it is done in Carroll §3.6).

Given an arbitrary metric in an arbitrary ind. system, we can always determine whether it represents flat spacetime (and in some funny coordinates), by calculating the Riemann tensor.
3.7 Properties of the Riemann Tensor

\( R_{\sigma\mu} \) has \( \nu^4 = 4^4 = 256 \) components. Because of symmetries, the number of independent components is much less. Already noted \( R^\sigma{}_{\mu\nu} = -R^\sigma{}_{\nu\mu} \), but there are others.

Symmetry (antisymmetry) of \( \text{Riemann} \) is a \( \text{cd} \)-independent property.

Lower to first order, \( R_{\sigma\mu} = 3\chi R_{\sigma\mu} \), and go to locally imbeddable (e.g., Riemann normal) cd's at point \( p \Rightarrow \Gamma^\nu_{\mu\nu}(p) = 0 \) \((\text{but } \Gamma^\alpha_{\mu\nu \beta}(p) \neq 0)\)

\[ R_{\sigma\mu}(p) = \� g_\chi (\partial_\nu \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\nu}) \]

\[ = \frac{1}{2} g_\chi \left\{ \partial_\mu \left[ g_\chi (\partial_\sigma g_{\nu\nu} + g_{\nu\nu,\sigma} - g_{\nu\nu,\sigma}) \right] - \partial_\sigma \left[ g_\chi (g_{\chi \mu} + g_{\mu \chi} - g_{\mu \chi}) \right] \right\} \]

\[ = \frac{1}{2} \left( g_{\chi \sigma \mu} + g_{\mu \chi \sigma} - g_{\nu \nu, \sigma} - g_{\nu \nu, \mu} - g_{\chi \mu, \sigma} + g_{\chi \mu, \mu} \right) \]

\[ = \frac{1}{2} \left( \partial_\mu \partial_\sigma \chi - \partial_\sigma \partial_\mu \chi - \partial_\mu \partial_\chi - \partial_\sigma \partial_\chi \right) \]

This is:
1) antisymmetric in \( \sigma, \chi \)
2) antisymmetric in \( \mu, \nu \)
3) symmetric under \( \sigma \leftrightarrow \mu \)

These symmetries leave \( R_{\sigma\mu\chi\nu} \), \( \frac{7 \times 6}{2} = 21 \) independent components

Symmetry in any index has \( \frac{n(n+1)}{2} \) independent components.

4) \( R_{\sigma\mu\chi\nu} + R_{\sigma\chi\nu\mu} + R_{\sigma\nu\mu\chi} = 0 \) (cyclic property in last 3 indices)

or, \( R_{\sigma[\mu,\nu]} = 0 \Rightarrow R_{[\sigma\mu\nu]} = 0 \)

Can also write \( R_{\sigma[\mu,\nu]} = 0 \) cyclic identity

This 4th condition gives 1 of the 21 components in terms of the other 20

\[ \therefore \text{The Riemann tensor has 20 independent components} \]
The Riemann tensor also satisfies a differential identity, known as the Bianchi identity:

\[ R^\kappa_{\beta\delta\epsilon\kappa} + R^\kappa_{\beta\epsilon\delta\kappa} + R^\kappa_{\epsilon\beta\delta\kappa} = 0 \]

\( \text{cyclic in } \kappa \text{ last 3} \)

**Proof:** Go again to locally trivial coordinate at point \( p \)

\[
R^\kappa_{\beta\delta\epsilon\kappa} = (\partial_\delta \Gamma^\alpha_{\beta\epsilon} - \partial_\epsilon \Gamma^\alpha_{\beta\delta} + \Gamma^\alpha_{\lambda\delta} \Gamma^\lambda_{\beta\epsilon} - \Gamma^\alpha_{\lambda\epsilon} \Gamma^\lambda_{\beta\delta})_{\beta\delta\epsilon\kappa}
\]

\[
= (\partial_\delta \partial_\epsilon \Gamma^\alpha_{\beta} - \partial_\epsilon \partial_\delta \Gamma^\alpha_{\beta} + \partial_\delta \Gamma^\alpha_{\beta} \partial_\epsilon - \partial_\epsilon \Gamma^\alpha_{\beta} \partial_\delta)_{\beta\delta\epsilon\kappa}
\]

\[
= \partial_\delta \partial_\epsilon \Gamma^\alpha_{\beta} - \partial_\epsilon \partial_\delta \Gamma^\alpha_{\beta} + \partial_\delta \Gamma^\alpha_{\beta} \partial_\epsilon - \partial_\epsilon \Gamma^\alpha_{\beta} \partial_\delta
\]

\( \text{evaluate at } p \)

\( \text{vanish at } p \)

\( \text{terms at type } \Gamma^\alpha_{\beta\gamma} \)

\( \text{vanish at } p \)

Thus \( (R^\kappa_{\beta\delta\epsilon\kappa} + R^\kappa_{\beta\epsilon\delta\kappa} + R^\kappa_{\epsilon\beta\delta\kappa})(p) = 0 \)
Decomposition of a tensor.

The decomposition of a tensor into its components is coordinate-dependent, e.g.

\[ T^{\mu\nu} \]

has a time-time part \( T^{00} \), time-space & space-time parts \( T^{0i}, T^{i0} \) and

space-space part \( T^{ij} \) but these transform into each other in ind. transformations.

The decomposition of a tensor into its symmetric and antisymmetric parts

\[ T^{\mu\nu} = T^{(\mu\nu)} + T^{[\mu\nu]} \]

is ind-independent.

Since symmetric \( S(y, z) = S(z, y) \) and antisymmetric \( A(y, z) = -A(z, y) \) are

coordinate-independent properties.

We can also separate out the trace of a tensor (often denoted by the same letter)

\[ T = g^{\mu\nu} T_{\mu\nu} \]

(only the symmetric part contributes)

and the trace-free part \( \hat{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \) of the symmetric part.

\[ T_{\mu\nu} = \frac{1}{n} T g_{\mu\nu} + \hat{T}_{\mu\nu} + T^{[\mu\nu]} \]

Higher-order tensors have more (in different ways) symmetric and antisymmetric parts
and more traces (contractions of \( g^{\mu\nu} \) different pairs of indices).
Decomposition of the Riemann tensor.

The symmetric $R^g_{\sigma\mu\nu} \equiv R_{\mu\nu}^\sigma$ already has produceed, separation into symmetric/antisymmetric parts.

Consider the traces: Because of antisymmetry

$g^{\sigma\alpha} R_{\sigma\alpha\mu\nu} = g^{\mu\nu} R_{\sigma\mu\nu} = 0$

We have one independent nonzero contraction, called the Ricci tensor

$R_{\mu\nu} \equiv R_{\alpha\mu\nu} = -R_{\mu\nu}^{\alpha\alpha}$

If it is symmetric, $R_{\nu\mu} = +R_{\mu\nu}$ and its trace

$R \equiv R_{\mu}^{\mu} = g^{\nu\mu} R_{\nu\mu}$ is called the curvature scalar (or Ricci scalar)

- The trace-free part of the Ricci tensor, $\hat{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu}$, rarely appears by itself.
- The trace-free part of the Riemann tensor is called the Weyl tensor

$$C_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} - \frac{2}{n-2} \left( g_{\mu\rho} R_{\nu\sigma} - g_{\nu\rho} R_{\mu\sigma} \right) + \frac{2}{(n-1)(n-2)} g_{\mu\rho} g_{\nu\sigma} R,$$

constructed so that it has the same symmetries as Riemann, but all possible traces disappear. Weyl is only defined for $n \geq 3$ dimensions, and for $n=3$ it $n = 0$.

Ricci has 10, and Weyl has 8 other 10, at Riemann's 20 degrees of freedom.
• Contract the Bianchi identity twice (IMPORTANT!):

\[ R^\alpha_{\beta\delta\epsilon} + R^\alpha_{\rho\delta\epsilon} + R^\alpha_{\rho\epsilon\delta} = 0 \]

\[ \Rightarrow R^\alpha_{\beta\delta\epsilon} - R^\alpha_{\rho\delta\epsilon} + R^\alpha_{\rho\epsilon\delta} = 0 \quad | \quad g^\alpha\epsilon \]

\[ \Rightarrow R^\epsilon_{\delta\epsilon} - R^\epsilon_{\rho\delta} + R^\alpha_{\rho\epsilon\delta} = 0 \]

\[ \Rightarrow 2R^\epsilon_{\delta\epsilon} = R^\epsilon_{\rho\delta} \]

\[ \Rightarrow \nabla^\epsilon R^\epsilon_{\delta\epsilon} = \frac{1}{2} \nabla^\rho R_{\rho\delta} \quad (\nabla^\epsilon = g^{\epsilon\mu} \nabla_\mu) \]

This means that if we define the Einstein tensor \( g_{\mu\nu} \) (in 4 dimensions):

\[ g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \]

also symmetric, \( g_{\mu\nu} = g_{\nu\mu} \)

its divergence vanishes

\[ g_{\mu\nu} = (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu})_{\rho\nu} = R_{\mu\nu\rho\nu} - \frac{1}{2} g_{\mu\nu} R_{\rho\nu} \]

\[ = R_{\mu\nu\rho\nu} - g_{\mu\nu} (R_{\rho\epsilon}^\epsilon) = R_{\mu\nu\rho\nu} - R^\epsilon_{\rho\epsilon} = 0 \]

\[ ) \]

\[ ) \]

\[ * ) \]

Note also that \( g^{\mu\nu} g_{\mu\nu} = g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} = -R. \] That is, \( g_{\mu\nu} \) is the "trace-reversed" \( R_{\mu\nu} \), i.e.,

\[ E^m_{\mu\nu} = g_{\mu\nu} \]

instead of \( \hat{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \) we would and defined \( g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \).
In 3-dim Euclidean geometry we have Gauss's theorem:

\[ \int_{\Sigma} \nabla \cdot \mathbf{V} \, d\Sigma = \int_{\partial \Sigma} \mathbf{V} \cdot d\mathbf{s} \]

In flat 4-dim spacetime the corresponding result is

\[ \int_{\hat{\Sigma}} d^4 \mathbf{x} \, \varepsilon_{\mu \nu} \mathbf{V}^\mu \mathbf{V}^\nu = \int_{\partial \hat{\Sigma}} d^3 s \, n^\mu \mathbf{V}^\mu \]

where \( n^\mu \) is the unit vector normal to the "surface" element \( d^3 s \),

pointing inward when timelike
outward when spacelike

In curved \( n \)-dim spacetime we have Stokes' Theorem:

\[ \int_{\mathcal{X}} \nabla_{\mu} \Gamma^\mu_{\nu\rho} \sqrt{|g|} \, d^n \mathbf{x} = \int_{\partial \mathcal{X}} \eta_{\mu\nu\rho} \sqrt{|g|} \Gamma^\mu_{\nu\rho} \, d^{n-1} \mathbf{x} \]

(assuming Christoffel connections)

where \( \gamma_{\mu\nu} \) is the induced metric on \( \partial \mathcal{X} \)

For example, if \( \mathbf{x} = \text{const} \) along a portion of \( \partial \mathcal{X} \), the metric

\[ ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2) \]

gives the induced metric

\[ ds^2 = -dt^2 + a(t)^2 (dy^2 + dz^2) \]

on \( \partial \mathcal{X} \)