4. GRAVITATION

Physics of gravitation: 1) How the "gravitational field" influences behaviour of matter
2) How matter determines the gravitational field

In Newtonian gravity:
\[ \ddot{a} = -\nabla \phi \]
\[ \nabla^2 \phi = 4\pi G \Sigma \]

4.1 Physics in Curved Spacetime

In GR, gravity appears as curvature of spacetime. Physical laws are generalized from flat to curved spacetime with no simple recipe

1) Take a law of physics, valid in inertial refs in flat spacetime
2) Write it in a coordinate-invariant (tensorial) form
3) Assert that the resulting law remains true in curved spacetime

In practice, this means replacing
\[ y_{\mu} \rightarrow g_{\mu \nu} y^\nu \]
\[ \Theta_{\mu} \rightarrow \Theta^\nu_{\mu} \]
\[ (\ddot{\tau} \rightarrow \ddot{\tau}) \]

so that the equations are tensorial in arbitrary refs and also in curved spacetime

This can be formalized as the Principle of General Covariance:

A physical eq. of GR is generally true in all ref. systems. If (a) the eq. is a tensor eq. (preserves its form under general ref. transformations) and (b) the eq. is true in SR.

(This is really just a consequence of the Equivalence Principle, and the requirement that physical laws do not depend on the ref. used to describe them.)
As an example of how this works, consider the motion of a free particle. In flat spacetime the motion of a free particle is given by Newton's 1st law: $\dot{a} = 0$.

We can write this as:

$$\frac{d^2x^i}{d\theta^2} = 0 \quad (1)$$

for the parametrized spacetime path $x^i(\theta)$.

To see how this corresponds to $\dot{a} = 0$:  
$$\frac{d^2x}{d\theta^2} = 0 \Rightarrow \frac{dx}{d\theta} = \text{const} = A$$
$$\frac{d^2y}{d\theta^2} = 0 \Rightarrow \frac{dy}{d\theta} = \text{const} = B$$

$$\frac{dx}{d\theta} = \frac{B}{A} = \text{const}, \text{ likewise } \frac{dy}{d\theta} = \text{const} \Rightarrow \text{the path is a straight line}$$

$$\frac{d^2t}{d\theta^2} = 0 \Rightarrow \frac{dt}{d\theta} = \text{const} \Rightarrow t = CN + D, \text{ i.e. } A \text{ and } t \text{ are affinely related}$$

$$\frac{dx}{d\theta} = \frac{B}{A} = \text{const}, \text{ likewise } \frac{dy}{d\theta} = \text{const} \Rightarrow \text{the path is a straight line}$$

$$\frac{d^2x}{d\theta^2} = 0 \quad (1)$$

Eq. (1) is not a tensor equation in general coordinates: While $\frac{dx^i}{d\theta}$ are the components of a vector, $\frac{d^2x^i}{d\theta^2}$ are not. Using the chain rule:

$$\frac{d^2x^i}{d\theta^2} = \frac{dx^i}{d\theta} \frac{dx^k}{d\theta} \Theta_{ik}$$

where $\Theta_{ik}$ is a tensor.

To generalize this eq. to curved spacetime, replace $\Theta_{ik} \rightarrow \Gamma_{ik}^j$ (in flat spacetime (and initial vel's) $\Theta_{ik} \rightarrow \Gamma_{ik}$) to get the GR version of Eq. (1),

$$\frac{dx^i}{d\theta} \Gamma_{ik}^j \left( \frac{dx^k}{d\theta} \right) = \frac{d^2x^i}{d\theta^2} + \nabla_i \frac{dx^k}{d\theta} \frac{dx^k}{d\theta} = 0, \quad (2)$$

the geodesic equation. Thus from the principle of general covariance follows that free (now called "freely falling") particles move along geodesics.
Another example is the conservation of energy and momentum, which in flat spacetime is given in differential form as

\[ \partial_{\mu} T^{\mu} = 0 \]  

(3)

The total energy and momentum of a system are

\[ E = \int T^{00} \, dv, \quad p^i = \int T^{0i} \, dv \]  

(in flat spacetime)

In flat spacetime one can integrate (3) to get for an isolated system

\[ \frac{dE}{dt} = 0, \quad \frac{dp^i}{dt} = 0. \]  

(4)

The generalization of (3) to curved spacetime is of course

\[ \nabla_{\mu} T^{\mu} = 0 \]  

(5)

This is called the energy continuity equation. We don't talk about conservation, since (5) cannot, in general, be integrated to get (4), for a curved spacetime. This may come as a shock to you, but in GR energy and momentum are not conserved! To understand why this is so, remember that in Newtonian physics we have the concept of gravitational potential energy. The kinetic energy of a particle in a gravitational field is of course not conserved by itself. In field theory, potential energy is replaced by the energy at the field. We have not put the energy of the gravitational field in \( T^{\mu\nu} \). It turns out, and in general, one cannot define an energy-momentum tensor for the gravitational field in GR.

In a deeper level, the conservation of energy and momentum are related to time and space translation invariance. A curved spacetime does not necessarily have such invariances. We also do not have the corresponding conservation laws.
Let us return to the geodesic equation (2). We have claimed that the motion of a freely falling particle (i.e., one that experiences no force, except gravity, which we refuse to call a force) satisfies it. This is our starting, following from our first principles. Does it have anything to do with reality — gravity as we know it from experience?

The Newtonian eq. for motion under gravity,

\[ \ddot{\alpha} = -\nabla \phi \]  

(6)

seems to work well in our everyday experience, e.g., in the solar system. Therefore, if eq. (2) is true, it should give (6) in some limit, appropriate for our circumstances.

Therefore, consider (2) in the Newtonian limit:

1) All particles move slowly (compared to the speed of light): \( v \ll c \).
2) The gravitational field is weak, i.e., the spacetime can be considered as a small perturbation to Minkowski space.
3) The gravitational field is static.

Condition 2) means that a coordinate system exists, where

\[ g_{\mu\nu} = \eta_{\mu\nu} + \eta_{\mu\nu}, \text{ and } |\eta_{\mu\nu}| < 1 \]  

(7)

In this coordinate system the meaning of coordinates \( t, \mathbf{x} \) is "close" to invariance of events in Minkowski space. Therefore

\[ \Rightarrow \frac{dx_i}{dt} \ll 1 \Rightarrow \frac{dx_i}{dt} \ll \frac{dl}{dt} \text{ and we can drop the small terms from (2), to get } \]  

\[ \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_\nu_\sigma \frac{dx^\nu}{dt} \frac{dx^\sigma}{dt} = 0 \]  

(8)

Condition 3) means that (a real system exist which, in addition to (7)) \( g_{\mu\nu} = 0 \).

This

\[ \Gamma^\mu_\nu_\sigma = \frac{1}{2} g^\mu_\rho (\partial_\nu g_{\rho\sigma} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma}) = -\frac{1}{2} g^\mu_\rho \partial_\nu g_{\rho\sigma} \]  

(9)

*) Since we are considering slowly moving particles, their path is timelike, and we can use the proper time \( \tau \) as our affine parameter.
In (9) we used the inverse metric $g^{\mu \nu}$. Clearly it should be close to the inverse Minkowski metric, $g^{\mu \nu} = \eta^{\mu \nu} + x^{\mu \nu}$, where $|x^{\mu \nu}| \ll 1$. Solve $x^{\mu \nu}$ from

$$
\delta^{\mu \nu} = g^{\mu \rho} g_{\rho \sigma} = (\eta^{\mu \rho} + x^{\mu \rho}) (\eta_{\rho \sigma} + x_{\rho \sigma}) = \delta^{\mu \nu} + \eta^{\mu \rho} x_{\rho \sigma} + \eta_{\rho \sigma} x^{\mu \rho} + x^{\mu \rho} x_{\rho \sigma}
$$

What we are doing here, is 1st order perturbation theory (something that we will later do a lot, in connection with gravitational waves and cosmological perturbation theory).

The basic idea is that we have some quantities which are small, called "1st order small". If we have the product of two such small quantities, it is then really small, called "2nd order small", so that we can ignore it. Thus we immediately drop from our equations all terms which are 2nd (or higher) order small.

In the above eq. $x^{\mu \rho} h_{\rho \sigma}$ is 2nd order, and we drop it. Thus

$$
\eta^{\mu \rho} x^{\rho \sigma} = - \eta^{\mu \rho} h_{\rho \sigma}
$$

$$
\Rightarrow \quad x^{\mu \rho} h_{\rho \sigma} = - \eta^{\mu \rho} h_{\rho \sigma} = - h_{\mu \sigma},
$$

where $h_{\mu \sigma} = g^{\mu \rho} g_{\rho \sigma} h_{\rho \sigma} = (\eta^{\mu \rho} + x^{\mu \rho}) (\eta_{\rho \sigma} + x_{\rho \sigma}) h_{\rho \sigma} = \eta^{\mu \rho} h_{\rho \sigma}$, which we dropped 2nd and 3rd order quantities in the last equality.

Note how in perturbation theory (around Minkowski metric) one can raise and lower indices at small quantities with $\eta_{\mu \nu}$, although the metric is $g_{\mu \nu}$.

Thus we get

$$
g_{\mu \rho} = \eta^{\mu \rho} - h_{\mu \rho}
$$

and

$$
\Gamma^{\mu}_{\alpha \beta} = - \frac{1}{2} g^{\mu \rho} \partial_{\alpha} g_{\beta \rho} = - \frac{1}{2} (\eta^{\mu \rho} - h_{\mu \rho}) \partial_{\alpha} h_{\beta \rho} = - \frac{1}{2} \eta^{\mu \rho} \partial_{\alpha} h_{\beta \rho} \quad \text{(11)}
$$

(We include this in our definition (condition 2) at Newtonian limit)

*) Here we assumed that also the derivatives $\partial_{\alpha} h_{\beta \rho}$ are small. In principle the smallness of $h_{\mu \rho}$ does not imply the smallness of $\partial_{\alpha} h_{\beta \rho}$. See T. V. R. Khanna & S. D. M. H. 827-9, how they can include rotating reference frames by allowing the $\partial_{\alpha} h_{\beta \rho}$ to be large.
The geodesic eq. (8) is now
\[ \frac{d^2 x^m}{d \tau^2} = \frac{1}{2} \eta^{mn} \partial_n \eta_{\alpha \beta} \frac{\partial \xi^\alpha}{\partial x^m} \frac{\partial \xi^\beta}{\partial x^n} \]
\[ \tau = 0: \quad \frac{d^2 t}{d \tau^2} = \frac{1}{2} \eta^{0\alpha} \partial_\alpha h_{00} \frac{\partial \xi^\alpha}{\partial t} = \frac{1}{2} \eta^{0\alpha} \partial_\alpha \eta_{00} \frac{\partial \xi^\alpha}{\partial t} = 0 \]
\[ \Rightarrow \frac{dt}{d\tau} = \text{const.} \quad \text{Dividing (12) by this const.} \]
\[ \tau = i: \quad \frac{d^2 x^i}{d \tau^2} = \frac{1}{2} \eta^{0i} \partial_0 h_{00} = \frac{1}{2} \eta^{0i} \partial_0 \eta_{00} = \frac{1}{2} \partial_i h_{00} \]

Now this is eq. (6), \( \tau = -\nabla \phi \), if we identify
\[ h_{00} = -2\phi \quad \text{or} \quad g_{00} = -(1+2\phi) \]

The Newtonian limit of the geodesic equation agrees with Newtonian gravity.

Of course we still need to find the field equation of GR, which states how a source of gravity determines the metric \( g_{\mu \nu} \), and show that \( \phi \) is a Newtonian potential. This is discussed in the next section.
4.2 Einstein Equation

Just as Maxwell's equations govern how the electric and magnetic fields respond to changes and currents, Einstein's field equation governs how the metric responds to energy and momentum. [Carroll].

I shall give two "derivations" of the Einstein equation: 1) Following how Einstein deduced it, by trial and error (this section) 2) From a variational principle, following Hilbert (section 4.3).

We need a replacement for the Poisson equation

$$\nabla^2 \phi = 4 \pi G \rho$$ \hspace{1cm} (\(\nabla^2 = g^{\mu \nu} \partial_\mu \partial_\nu\)) \hspace{1cm} (1)

of Newtonian gravity. Both sides need to be tensors. The mass density \(\rho\) is not a tensor (volume elements are Lorentz-contradicted in boosts). The proper tensor generalization is the energy tensor \(T_{\mu \nu}\). In §4.1 we saw that \(\phi\) corresponds to (a perturbation in) a metric component. Thus we are looking for an equation of the type

$$\left[\nabla^2 g_{\mu \nu}\right] \propto T_{\mu \nu}$$

where \(\nabla^2\) represents some second-order tensor, which should involve second derivatives of the metric. What could \(\nabla^2\) be? The obvious generalization of the Laplacian \(\nabla^2\) to 4-dim curved spacetime, the \(\delta\)-Laplacian, of the metric

$$\nabla^2 g_{\mu \nu} \propto \nabla^2 g_{\mu \nu}$$

won't do, since it is always zero \((\nabla^2 g_{\mu \nu} = 0)\).

We have already met the obvious candidate, the Riemann tensor \(R^\lambda_{\mu \rho \nu}\), which describes the curvature of spacetime, and is made out of \(\delta\) (first and) second derivatives of the metric. It is of wrong order, but if we contract it, we get

$$R_{\mu \nu} = R^\lambda_{\mu \lambda \nu}$$

a symmetric second-order tensor, just like \(T_{\mu \nu}\).
So here about

$$R_{\mu\nu} = \nabla_{\mu} T_{\nu}$$

(2)

(what $\nabla$ is a proportionality constant, a constant of nature describing the strength of the gravitational interaction, to be determined by experiment, or by comparison to eq. (1)).

This was, in fact, Einstein’s first proposal, but we can see that it leads to nonsense:

We know that $\nabla^M T_{\mu\nu} = 0$. Therefore we would also have $\nabla^M R_{\mu\nu} = 0$.

But Bianchi’s eq. says that $\nabla^M R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R$.

Taking the trace of (2) would give $R = \nabla T$, so that we would have

$$\nabla T = 0 \Rightarrow T = T^M = \text{const}.$$ But in vacuum $T_{\mu\nu} = 0 \Rightarrow T = 0$$

which for matter $T \neq 0$.

For perfect fluid $T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} + p g_{\mu\nu}$

$$\Rightarrow T = T^M = (\rho + p) u_{\mu} u_{\nu} + p g_{\mu\nu} = \rho + p = -\rho_0 + p$$

so for dust ($p = 0$), $T = \rho_0$.

(For radiation ($p = \frac{1}{3} \rho$), $T = -\rho + \rho_0 = 0$, so we can have $T = 0$ for non-vacuum, too).

We have already encountered the covariant tensor, the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu},$$

which was constructed so that the Bianchi eq. for it is $\nabla^M G_{\mu\nu} = 0$. Thus the Bianchi eq. leads to no problems, but instead guarantees that $G_{\mu\nu}$ satisfies the same eq. as $T_{\mu\nu}$. Thus the field equation of GR, the Einstein equation, is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \nabla T_{\mu\nu}$$

(3)

Einstein himself found this combination, $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, by trying out

$$\alpha R_{\mu\nu} + \beta g_{\mu\nu} = T_{\mu\nu}.$$ Other values, except $\frac{\alpha}{\beta} = -\frac{1}{2}$ lead to similar problems as above.
We justified the LHS of the Einstein equation, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$, by saying that "it leads to no problems". But maybe there are other possibilities. Well, there are other possibilities, but they are more complicated.

It turns out that $G_{\mu\nu}$ is unique in the sense that it is the only tensor (up to a multiplicative constant) which satisfies the following requirements:

1) $G_{\mu\nu} = 0$ for a flat spacetime

2) It is constructed from the Riemann tensor and the metric, and from nothing else

3) It is linear in the Riemann tensor

4) It is symmetric and of second order

5) It automatically satisfies $\nabla^\mu G_{\mu\nu} = 0$

(like 2 and 3 are simplicities, if you chop them, you have available a host of even more complicated alternative gravity theories).

Thus GR is the simplest theory that you can construct from the idea that gravity is curvature in spacetime.

- Contracting (3) we get $R^\mu_\mu - \frac{1}{2}R = \mu T^\mu_\mu$, or

$$-\mathbf{R} = \mu \mathbf{T} \quad (4)$$

which allows us to write the Einstein eq. (3) in the alternative form

$$R_{\mu\nu} = \mu (T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}) \quad (5)$$

This is useful, since we see immediately that

$$R_{\mu\nu} = 0 \quad \text{in vacuum} \quad (T_{\mu\nu} = 0) \quad (6)$$
Note that other components of the Riemann tensor (i.e., the Weyl tensor) may be nonzero in vacuum. That is, spacetime can be (and usually is) curved outside the sources, too. This we, of course, expect from a theory of gravity: that the gravitational "field" at a body extends into the surrounding empty space.

To summarize:

1) In flat spacetime, $R^\alpha_{\beta\nu\sigma} = 0$

2) In empty spacetime (vacuum), $R_{\mu\nu} = G_{\mu\nu} = 0$

A flat region of spacetime must be empty, but an empty region is usually not flat.
• We shall have to show that the Einstein eq. (3) agrees with (1) in the Newtonian limit and determines the constant \( \Lambda \).

• Assume the perfect fluid form \( T_{\mu \nu} = (\epsilon + p)u_\mu u_\nu + pg_{\mu \nu} \) and go to the Newtonian limit. The pressure \( p \) is due to thermal motions at particles. If they are \( \ll 1 \), then \( p \approx 0 \). Thus we can approximate \( T_{\mu \nu} = g_{\mu \nu} \).

The LHS of (3) involves derivatives at the metric, for which only the perturbation \( h_{\mu \nu} \) contributes. Therefore the LHS is first order small \( \Rightarrow \) the RHS is also first order small \( \Rightarrow \) \( g \) in first order small \( \Rightarrow \) we need \( u_\mu \) only to 0th order, while

\[
\Gamma^\mu_{\nu \lambda} = \left( 1, \frac{\partial}{\partial x}, 0 \right), \quad \Gamma^\mu_{\nu \lambda} = \left( 1, 0, 0 \right) \quad (0th \ \text{order})
\]

\( \Rightarrow T_{oo} = 0, \quad T_{ij} = T_{oi} = 0 \) \( \Rightarrow T = g^{\mu \nu} T_{\mu \nu} = g^{oo} T_{oo} = 0oo T_{oo} = -8
\)

The 00-components of (5) now says

\[
R_{oo} = h (T_{oo} - \frac{1}{2} T_{oo}) = h (0 - \frac{1}{2} 0) = \frac{1}{2} h_8
\]

Express \( R_{oo} \) in terms of the metric.

\[
R_{oo} = R^\lambda_{0 \lambda 0} = R^i_{oo i} \quad (\text{since } R^\lambda_{\mu \lambda 0} = 0 \text{ from antisymmetry})
\]

and

\[
R^i_{oo i} = \frac{\partial^2 f_{oo}}{\partial x^i \partial x^i} - \frac{1}{2} \frac{\partial f_{oo}}{\partial x^i} f_{oo} + \frac{1}{2} \frac{\partial f_{oo}}{\partial x^i} f_{oo} - \frac{1}{2} f_{oo} \frac{\partial^2 f_{oo}}{\partial x^i \partial x^i}
\]

2nd order small, since \( f_{oo} \) in 1st order

Thus \( R_{oo} = \frac{1}{2} \frac{\partial^2 f_{oo}}{\partial x^i \partial x^i} = -\frac{1}{2} \frac{\partial f_{oo}}{\partial x^i} (g^{ii} g_{oo} g_{oo}) = -\frac{1}{2} \frac{\partial f_{oo}}{\partial x^i} (g^{ii} g_{oo}) \)

\[
= -\frac{1}{2} \frac{\partial f_{oo}}{\partial x^i} (g^{ii} g_{oo}) = -\frac{1}{2} \nabla^2 h_{oo}
\]

Thus (7) becomes \( -\frac{1}{2} \nabla^2 h_{oo} = \frac{1}{2} h_8 \), or

\[
\nabla^2 h_{oo} = -h_8
\]

*) in the cut system that is "close" to Minkowski inertial cut's

**) off-diagonal components are second-order small
Comparing (8), $\nabla^2 h_{\alpha\beta} = -x g_{\alpha\beta}$, to (1), $\nabla^2 \phi = 4\pi G\phi$, we see that in the Newtonian limit we get the relation

$$h_{\alpha\beta} = -\frac{x}{4\pi G} \phi$$

between the metric perturbations and the Newtonian potential. For free fall motion to agree with Newtonian theory in the Newtonian limit we needed

$$h_{\alpha\beta} = -2\phi$$

Therefore we conclude that the correct value for the proportionality constant is

$$\mu = 8\pi G,$$

and we can finally write the Einstein equation as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

or in the more compact form*

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Also, do not forget the form

$$R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu})$$

*) Sometimes "geometric" units are used, where $G=1$, or $8\pi G=1$. Then we have just $G_{\mu\nu} = T_{\mu\nu}$.
4.3 Lagrangian Formulation

4.3.1 Field Theory in Flat Spacetime

Fields $\phi^a(x^\mu) = \phi^a(t, x_1)$, $a = 1, \ldots, N$

- Principle of Least Action: For fixed initial $(t = t_1)$ and final $(t = t_2)$ field configurations $(\phi^a(x))$, the field evolves from $t_1$ to $t_2$ so as to extremize the action

$$S = \int_{t_1}^{t_2} dt \mathcal{L}, \quad \mathcal{L} = \int d^3x (\mathcal{L}[\phi^a, \partial_t \phi^a]) d^3x$$

Logarithmic function

Theory is given by specifying $\mathcal{L}$

1. Vary $\phi^a(x) \rightarrow (\phi^a(x) + \delta \phi^a(x)) \Rightarrow \mathcal{E}_a \delta \phi^a + \partial_t \phi^a + \partial_t (\delta \phi^a)$

$$\delta S = \int d^3x \left[ \frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \partial_t \delta \phi^a \right] = 0 \quad \forall \delta \phi^a$$

integrate by parts

$$\mathcal{E}_a \left[ \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \delta \phi^a \right] = \mathcal{E}_a \left[ \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \right] \delta \phi^a + \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \partial_t \delta \phi^a$$

$$\delta S = \int d^3x \{ \mathcal{E}_a \delta \phi^a - \mathcal{E}_a \left[ \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \right] \} \delta \phi^a + \int d^3x \mathcal{E}_a \left[ \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \right] \delta \phi^a$$

2. The latter term

$$\frac{1}{t_2 - t_1} \int d^3x \left[ \int d^2z \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \right] \delta \phi^a + \int d^3x \left[ \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \right] \delta \phi^a = (y) + (z)$$

$$\frac{1}{t_2 - t_1} \int d^3x \left[ \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \right] \delta \phi^a + \int d^2z \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \delta \phi^a = 0$$

assuming $\delta \phi^a$ vanishes on boundary $\partial \Sigma$

$$\mathcal{E}_a \left[ \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \right] = 0$$

Euler–Lagrange equation

$$\mathcal{E}_a \left[ \frac{\partial \mathcal{L}}{\partial \partial_t \phi^a} \right] = 0$$

More spatial boundaries outside region of interest, who assume $\phi^a$, or at least $\delta \phi^a$, vanishes.
4.3.2 Field Theory in Curved Spacetime

In flat spacetime, the boundary term is

$$\oint_{\partial \Sigma} \mathbf{F} \cdot d\mathbf{S} = \int_{\Sigma} \nabla \cdot \mathbf{F} \, d\Sigma$$  

(Cauchy's Theorem)

In curved spacetime, this is replaced by

$$\oint_{\partial \Sigma} \mathbf{F} \cdot d\mathbf{S} = \int_{\Sigma} \nabla \cdot \mathbf{F} \, d\Sigma$$  

(Stokes' Theorem)

(assume Christoffel connection)

Consider the action $S(\phi)$,

$$L = L(\phi, \nabla \phi)$$

For $\phi^a$, define $\delta \phi^a \Rightarrow \nabla_\mu \phi^a \rightarrow \nabla_\mu (\delta \phi^a)$

\[
\delta S = \int_{\Sigma} \sqrt{-g} \, d^4x \left[ \frac{\partial L}{\partial \phi^a} \delta \phi^a + \frac{\partial L}{\partial (\nabla \phi^a)} \nabla_\mu (\delta \phi^a) \right]
\]

Now

$$\nabla_\mu \left[ \frac{\partial L}{\partial (\nabla \phi^a)} \delta \phi^a \right] = \nabla_\mu \left[ \frac{\partial L}{\partial (\nabla \phi^a)} \right] \delta \phi^a + \frac{\partial L}{\partial (\nabla \phi^a)} \nabla_\mu (\delta \phi^a)$$

and

$$\int_{\Sigma} \sqrt{-g} \, d^4x \nabla_\mu \left[ \frac{\partial L}{\partial (\nabla \phi^a)} \delta \phi^a \right] = \int_{\Sigma} \sqrt{-g} \, d^4x \nabla_\mu \left[ \frac{\partial L}{\partial (\nabla \phi^a)} \right] \delta \phi^a + \frac{\partial L}{\partial (\nabla \phi^a)} \nabla_\mu (\delta \phi^a)$$

by Stokes' Theorem.

$$= 0 \quad \text{assuming} \; \delta \phi^a \; \text{vanishes on boundary} \; \partial \Sigma$$

\[
\therefore \; \delta S = \int_{\Sigma} \sqrt{-g} \, d^4x \left[ \frac{\partial L}{\partial \phi^a} - \nabla_\mu \left( \frac{\partial L}{\partial (\nabla \phi^a)} \right) \right] \delta \phi^a
\]

Requiring $\delta S = 0$ for $\delta \phi^a$ leads to

$$\frac{\partial L}{\partial \phi^a} - \nabla_\mu \left( \frac{\partial L}{\partial (\nabla \phi^a)} \right) = 0 \quad \text{as the field equation}$$

*) The normal vector $n^\mu$ is taken to point inward, if timelike.

outward, if spacelike.
Example: Scalar field \( \phi \) :

\[
L = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi)
\]

\[
S = \int \left[ -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right] \sqrt{-g} \, d^4x
\]

\[
\frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial \phi}
\]

\[
\frac{\partial L}{\partial (\nabla_\mu \phi)} = -\frac{1}{2} g^{\mu\nu} \delta_\mu^\alpha \nabla_\nu \phi - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \delta_\nu^\alpha = -\frac{1}{2} \left( g^{\nu\mu} \nabla_\nu \phi + g^{\mu\nu} \nabla_\nu \phi \right)
\]

\[
= -\nabla^\phi \phi
\]

\[
\therefore \quad \frac{\partial L}{\partial \phi} - \nabla_\mu \left( \frac{\partial L}{\partial (\nabla_\mu \phi)} \right) = -\nabla_\nu \nabla^\nu \phi - \frac{\partial V}{\partial \phi} = 0
\]

or

\[
\Box^2 \phi - \frac{\partial V}{\partial \phi} = 0
\]

where \( \Box^2 = \nabla_\mu \nabla^\mu = g_{\mu\nu} \nabla_\mu \nabla_\nu \)
4.3.3 Varying the Spacetime Metric

\[ g_{\mu \nu} + \delta g_{\mu \nu} \]

Lagrangian? Since \( \nabla \delta g_{\mu \nu} = 0 \), it is useful for Lagrangian, and 2nd derivatives of \( g_{\mu \nu} \).

The only independent scalar constructed from the metric and its 1st and 2nd derivatives, which is not higher than 2nd order in them, is the scalar curvature \( R \). Take \( \mathcal{L} = R \)

\[ S_H = \int d^n x \sqrt{-g} R \]

\( \text{Hilbert action} \)

- It's more convenient to use variation of inverse metric \( \delta g^{\mu \nu} \)

\[ g^{\mu \lambda} g_{\nu \lambda} = \delta^\mu_\nu = \text{const} \implies \delta (g^{\mu \lambda} g_{\nu \lambda}) = g_{\nu \lambda} \delta g^{\mu \lambda} + g^{\mu \lambda} \delta g_{\nu \lambda} = 0 \mid \cdot \delta g_{\nu \lambda} \]

\[ \implies g_{\mu \nu} \delta g_{\mu \nu} = - \delta g_{\mu \nu} \]

We'll also use \( \delta \sqrt{-g} = - \frac{\delta g}{2\sqrt{-g}} = \frac{1}{2} \sqrt{-g} \left( g_{\mu \nu} \delta g^{\mu \nu} = -\frac{1}{2} \sqrt{-g} g_{\mu \nu} \delta g^{\mu \nu} \right) \)

**Used here:** \( \delta g = g g^{\mu \nu} \delta g_{\mu \nu} \) applied to inverse metric \( \delta g^\mu_\lambda = g^\mu_\lambda g^{\nu \mu} \delta g_{\nu \lambda} \)

Since \( \delta g^\mu_\lambda = -g^2 \delta g \), we have \( \delta g = -g g^{\mu \nu} \delta g_{\mu \nu} \)

\[ S_H = \int d^n x \sqrt{-g} \ g^{\mu \nu} \mathcal{R}_{\mu \nu} \]

\[ \delta S_H = \underbrace{\int d^n x \sqrt{-g} \ g^{\mu \nu} \delta \mathcal{R}_{\mu \nu}}_{\delta S_1} + \underbrace{\int d^n x \sqrt{-g} \ g_{\mu \nu} \delta g^{\mu \nu}}_{\delta S_2} + \underbrace{\int d^n x \mathcal{R} \delta \sqrt{-g}}_{\delta S_3} \]

- The work is in \( \delta S_1 \), i.e. \( \delta \mathcal{R}_{\mu \nu} \)

***) i.e. terms are products of at most two of them

***) The result \( \delta g = g g^{\mu \nu} \delta g_{\mu \nu} \) derived in same way as homework \( g_{\mu \nu} = g^{\mu \lambda} g g_{\lambda \nu} \)
\[ S_S = R_{\mu\nu} - \Theta_\alpha \Gamma^{\alpha}_{\mu\nu} + \Gamma^\gamma_{\mu\nu} \Gamma^\epsilon_{\gamma\mu} - (\Lambda \epsilon_{\mu\nu}) \]

\( S_I \) can be expressed in terms of \( S \), but consider first arbitrary \( S_I \)

\[ M^\alpha_{\mu\nu} \rightarrow M^\alpha_{\mu\nu} + \delta M^\alpha_{\mu\nu} \]

\( \delta M^\alpha_{\mu\nu} \rightarrow \) a difference between two covariants \( \Rightarrow \) a tensor

\[ \nabla_\alpha (\delta M^\alpha_{\mu\nu}) = \Theta_\alpha (\delta M^\alpha_{\mu\nu}) + \Gamma^\beta_{\mu\nu} \delta M^\beta_{\gamma\mu} - \Gamma^\gamma_{\mu\nu} \delta M^\gamma_{\omega\mu} - M^\gamma_{\mu\nu} \delta M^\gamma_{\omega\mu} \]

all covariant derivatives have taken wrt \( g_{\mu\nu} \), not \( g_{\mu\nu} + \delta g_{\mu\nu} \)

Now \( \delta R_{\mu\nu} = \Theta_\alpha (\delta M^\alpha_{\mu\nu}) + \delta M^\beta_{\mu\nu} \delta M^\beta_{\gamma\mu} + \Gamma^\gamma_{\mu\nu} \delta M^\gamma_{\omega\mu} - (\Lambda \epsilon_{\mu\nu}) \)

\[ = \nabla_\alpha (\delta M^\alpha_{\mu\nu}) - (\Lambda \epsilon_{\mu\nu}) \]

\[ \Rightarrow \delta R_{\mu\nu} = \delta \left( R^\alpha_{\mu\nu} \right) = \delta R^\alpha_{\mu\nu} = \nabla_\alpha (\delta M^\alpha_{\mu\nu}) - \nabla_\nu (\delta M^\alpha_{\mu\nu}) \]

\[ = \int d^3x \sqrt{-g} \left[ \nabla_\alpha (\delta M^\alpha_{\mu\nu}) - \nabla_\nu (\delta M^\alpha_{\mu\nu}) \right] \]

using metric compatibility

\[ = \int d^3x \sqrt{-g} \left[ g^{\mu\nu} \delta M^\alpha_{\mu\nu} - g^{\nu\mu} \delta M^\alpha_{\mu\nu} - g^{\nu\mu} \delta M^\alpha_{\mu\nu} \right] \]

\[ = 0 \]

by Stokes theorem

assuming the variation \( S_I \) vanishes at the boundaries.

- Altogether \( S_S = \int d^3x \sqrt{-g} \left[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] \delta g_{\mu\nu} \)

- Requiring \( S_S = 0 \) \( \forall \delta g_{\mu\nu} \) gives

\[ \frac{\delta S_S}{\delta g_{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \]

What about \( T_{\mu\nu} \)? We had no matter fields in our Lagrangian; thus we get the field eq. in vacuum.

- Suppose the field action is \( S = \frac{1}{16\pi G} S_S + S_M \) where \( S_M = \int d^3x \sqrt{-g} L(\phi^0, \phi^\alpha, \phi^\beta) \).

Requiring \( S_S = 0 \) gives

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left( \frac{\delta S_M}{\delta g_{\mu\nu}} \right) \]

Exercise: Show \( S_I^\mu = -\frac{1}{2} \left[ g_{\mu\nu} \nabla_\alpha (\delta g^\alpha_\nu) + g_{\nu\mu} \nabla_\beta (\delta g^\beta_\alpha) - g_{\mu\alpha} g_{\nu\beta} \nabla_\gamma (\delta g^{\gamma\beta}) \right] \)