8. COSMOLOGY

Cosmology = study of the universe as a whole, its structure and evolution.

Which solution of the Einstein equation corresponds to the universe we live in?

Clearly some approximation is needed. We are now interested in large scales.

In this chapter we discuss a 0th order approximation to the universe, which is
homogeneous and isotropic

- **Homogeneous**: all locations are equal (translation isometries)
- **Isotropic**: all directions are equal (rotation isometries)

Isotropy refers to isotropy at some point

A space may be homogeneous, but not isotropic

or isotropic at a certain point, but not homogeneous

Isotropic at every point → homogeneous

Homogeneous and isotropic at one point → isotropic at every point

- **Copernican principle**: We do not occupy a privileged position in the universe

- **Cosmological principle**: The universe is homogeneous and isotropic (at large scales)

There is strong observational evidence for the isotropy of our universe (galaxy
number counts, radio sources, diffuse X-ray and microwave backgrounds, and especially
the cosmic microwave background (rms temperature variation $3\times10^{-5}$)). This of
course refers to isotropy at our location.

The evidence also supports homogeneity at large scales, but is not as strong, since
our observations are all from a single location.

Isotropy at our location combined with Copernican principle

$\Rightarrow$ isotropic everywhere $\Rightarrow$ homogeneous $\Rightarrow$ Cosmological principle

Here isotropy and homogeneity refer to space (3 translation isometries and
3 rotation isometries), not spacetime (4 translation isometries and 6 rotation
isometries). It turns out that our universe is not homogeneous in time, but
it evolves.

For their theoretical importance we however consider first homogeneous and
isotropic (maximally symmetric) spacetimes.
8.1 Maximally Symmetric Spaces

A manifold has a symmetry if its geometry is invariant under a transformation that maps \( M \) to itself.

Symmetries of \( M \) are called isometries.

We shall avoid definitions and mathematical machinery (involving e.g. Killing vector fields), see Cornwell §3.8 and Appendix B, if you are interested.

Example: Minkowski space \( ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad \text{where} \quad \gamma_{\mu \nu} = \eta_{\mu \nu} \)

Isometries:

- Translating: \( \mathcal{X}^M \rightarrow \mathcal{X}^M + \mathcal{A}^M \) (4 parameters)
- Lorentz transforming: \( \mathcal{X}^M \rightarrow \mathcal{L}_{\mathcal{U}}^{\mathcal{V} \mathcal{X}} \) (6 parameters)

Minkowski space has 10 isometries.

An example of a space with maximal symmetry is \( \mathbb{R}^n \), with the Euclidean metric

\( ds^2 = \delta_{ij} dx^i dx^j \quad (i,j = 1, \ldots, n) \)

The isometries are translations and rotations.

Translations: \( n \) independent translations.

Rotations: For \( n \geq 3 \) there are 3 independent rotations, but this is an accident; only in \( n = 3 \) rotation is around an axis (in \( n = 2 \) it is around a point; only 1 independent rotation). In general, rotation is associated with the plane of rotation \( \Rightarrow \) the number of independent rotations = number of pairs of coordinates = \( \frac{n(n-1)}{2} \)

The number of isometries is \( n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} \)

(\( = 10 \) for \( n = 4 \))

This is the maximum number of isometries for an \( n \)-dimensional manifold

- Minkowski space is also maximally symmetric.

In general, the isometries are translations and rotations.

If the metric has Lorentzian signature, some of the rotations are boosts.
A maximally symmetric manifold need not be flat. But the curvature must be the same everywhere (homogeneity), due to translations isometries, and in every direction (isotropy), due to rotations isometries.

The most familiar examples are n-dimensional spheres $S^n$. They can be embedded in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, with metric

$$ds^2 = \sum_{i=1}^{n+1} x_i^2$$

where $x_i$ is the $n$-dim surface

$$\sum_{i=1}^{n+1} x_i x_i = \alpha^2$$

where $\alpha$ is the "radius" of the sphere.

The maximal symmetry constrains the curvature tensor to have a unique form:

In locally orthonormal coordinates (where $g_{\mu\nu} = \eta_{\mu\nu}$ at $p$) it must be zero and all tensors which are invariant under Lorentz transformations: $\eta_{\mu\nu}, \Sigma_{\mu\nu}, \Sigma_{\mu\nu\rho\sigma}$.

The only combination with the symmetries of Riemann is $\eta_{\mu\nu} \eta_{\mu\nu} - \eta_{\mu\nu} \eta_{\mu\nu} = g_{\mu\nu} g_{\mu\nu} - g_{\mu\nu} g_{\mu\nu}$

In general coordinates

$$R_{\mu\nu\rho\sigma} = H (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\nu} g_{\rho\sigma})$$

where $H$ is a constant

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = g^{\lambda\nu} R_{\mu\lambda\nu} = H (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\nu} g_{\rho\sigma}) = H (n-1) g_{\mu\nu}$$

and the curvature scalar is

$$R = g^{\mu\nu} R_{\mu\nu} = H (n-1) \delta_{\mu\nu} = H n(n-1)$$

$$\Rightarrow H = \frac{R}{n(n-1)}$$

The maximally symmetric spaces are classified by the signature of the metric, 1) the dimension $n$, and 3) the value of $R$.

(positive, negative, or zero)
• For the positive definite metric ("Euclidean signature") we already had the cases:
  flat, \( R = 0 : \mathbb{R}^n \)
  positively curved, \( R > 0 : S^n \)
  and for the Lorentzian signature:
  flat, \( R = 0 : \text{Minkowski space} \)

• For the positive definite metric (Euclidean signature), the
  negatively curved, \( R < 0 : H^n \)
  space, "the \( n \)-hyperbolic", can be embedded in the \( n+1 \)-dimensional
  Minkowski space, with metric
  \[
  ds^2_{(n+1)} = -du^2 + \sum_{ij} dx_i dx_j \quad (i,j = 1, \ldots, n)
  \]
  In this space we have the \( n \)-dim surfaces (instead of spheres)
  \[-u^2 + \sum_{ij} x_i x_j = -\alpha^2 \quad H^n \]
  and
  \[-u^2 + \sum_{ij} x_i x_j = +\alpha^2 \quad dS^n \quad (de Sitter space)\]

• Consider these for \( n = 2 \), for simplicity
  \[
  ds^2_{(3+1)} = -du^2 + dw^2 + dx^2
  \]

We note that the surface \( H^2 \) is
  spacelike, i.e., its metric will be
  positive definite; but the surface
  \( dS^2 \) is timelike, i.e., its metric will
  have Lorentzian signature.

\( H^n \) has negative constant curvature, \( R < 0 \)
\( dS^n \) has positive constant curvature, \( R > 0 \)
The 4-dimensional spacetime (Lorentzian signature) with constant positive curvature, $dS^4$, is called de Sitter space. It can be embedded in 5-dimensional Minkowski space, with metric

$$ds^2 = -dt^2 + dw^2 + dx^2 + dy^2 + dz^2$$

giving the surface

$$-u^2 + w^2 + x^2 + y^2 + z^2 = c^2$$

We can gain an understanding of de Sitter space by considering the version with two less dimensions (the "dS^2" of previous page), see Figure. Each circle in this figure corresponds to an $S^3$ in the full 4-dim de Sitter space. This space is maximally symmetric, i.e., it is homogeneous and isotropic. It may not look like that in the figure; but remember that the embedding space is Minkowski! Thus the tilted circle is the same size as the smallest horizontal circle, at its apparent "waist".

Figure. de Sitter space w 2 dimensions
(y, z) left out. (The u-axis is stretched compared to w and x-axes.)

*) Because of the Lorentzian signature, the one-dimensional axis of directions (spacelike, lightlike, timelike) is so small that isometry is now more limited.
Study de Sitter space, leaving 2 dimensions out. The full 4-dim case (which can be embedded in 5-dim Minkowski space) is similar, just more tedious.

The embedding space is completely rigid, but it is very helpful in visualising de Sitter space. It has now the metric

\[ ds_{(5)}^2 = -dt^2 + dw^2 + dx^2 \]

and the de Sitter space is embedded in it on the 2-surface

\[ -u^2 + w^2 + x^2 = \alpha^2 \]

Introduce coörd's \((t,u), -\infty < t < \infty, 0 < u < 2\pi\) in de Sitter space by

\[ \begin{align*}
  u &= \alpha \sinh \frac{t}{\alpha} \\
  w &= \alpha \cosh \frac{t}{\alpha} \cos u \\
  x &= \alpha \cosh \frac{t}{\alpha} \sin u
\end{align*} \]

\[ \begin{align*}
  du &= \cosh \frac{t}{\alpha} \, dt \\
  dw &= \sinh \frac{t}{\alpha} \cos u \, dt - \alpha \sinh \frac{t}{\alpha} \sin u \, du \\
  dx &= \sinh \frac{t}{\alpha} \sin u \, dt + \alpha \cosh \frac{t}{\alpha} \cos u \, du
\end{align*} \]

(You can easily check that \( -u^2 + w^2 + x^2 = \alpha^2 ( -\sinh^2 + \cosh^2 ) = \alpha^2 \).)

From this we get the metric of the de Sitter space:

\[ ds^2 = -dt^2 + dw^2 + dx^2 = \ldots = -dt^2 + \alpha^2 \cosh^2 \frac{t}{\alpha} \, dp^2 \]

In the full 4-dim case we define

\[ \begin{align*}
  u &= \alpha \sinh \frac{t}{\alpha} \\
  w &= \alpha \cosh \frac{t}{\alpha} \cos \phi \\
  x &= \alpha \cosh \frac{t}{\alpha} \sin \phi \sin \theta \\
  y &= \alpha \cosh \frac{t}{\alpha} \sin \phi \cos \theta \\
  z &= \alpha \cosh \frac{t}{\alpha} \sin \phi \sin \theta \sin \phi
\end{align*} \]

and arrive at the full metric of de Sitter space 2-sphere \( d\Omega_{(5)}^2 \)

\[ ds^2 = -dt^2 + \alpha^2 \cosh^2 \frac{t}{\alpha} \left[ d\phi^2 + \sin^2 \phi (d\theta^2 + \sin^2 \theta \, d\phi^2) \right] \]

So each circle \( \phi \in [0,2\pi) \) is replaced by a 3-sphere

This coordinate system covers the whole de Sitter space.
Other end, system on (2-chain) de Sitter space

Could use 2 of the 3 end's at the embedding space, e.g. \( w, x \)

But then we can only half of de Sitter space, since it has two values at \( u \) for each \( w, x \).

Replace \( u, w \) with \( u + w \) and \( u - w \). Now seem to do slightly better, since

\[-u^2 + w^2 + x^2 = 0^2 \implies (w + u)(w - u) = 0^2 - x^2 \implies w - u = \frac{x^2 - u^2}{w + u}\]

So if I pick \( \frac{\tilde{w}}{\tilde{w} + u} \) and \( x \) as my two \( w, u \)'s, then's only one point \((w - u, w + u, x)\) corresponding to \( u \) in de Sitter space.

Except for \( w + u = 0 \): This corresponds to two

sharply linear \( u = -w \), and \( x = \pm \infty \).

This also leads to two separate null directions \((\tilde{w}, x)\)

1) \( 0 < \tilde{w} < \infty \), \(-\infty < x < \infty \)

2) \(-\infty < \tilde{w} < 0 \), \(-\infty < x < \infty \)

Neither one uses the boundary at \( \tilde{w} = u + w = 0 \).

\[
\begin{align*}
w - u &= \frac{x^2 - u^2}{\tilde{w}} \\
w + u &= \tilde{w}
\end{align*}
\]

\[
\Rightarrow \quad ds^2 = -du^2 + dw^2 + dx^2 = \quad \quad = \quad -\left(\frac{x^2 - u^2}{\tilde{w}}\right) dt^2 - \frac{2\tilde{w}}{\tilde{w}} dt dx + dx^2
\]

This becomes much clearer by defining (for \( \tilde{w} > 0 \)) \( \tilde{x} = \alpha \ln \frac{\tilde{w}}{\tilde{x}} \), \( \tilde{x} = \frac{\tilde{x}}{\tilde{w}} \), \( \tilde{w} = \alpha \tilde{x} \tilde{e}^{\tilde{x}/\tilde{w}} \)

\[
\begin{align*}
\tilde{d}t &= \frac{\alpha}{\tilde{w}} \tilde{d}t \quad \tilde{d}x = \frac{\alpha}{\tilde{w}} \tilde{d}x - \frac{\alpha}{\tilde{w}} \tilde{d}t \tilde{d}x \\
\tilde{d}s^2 &= -\tilde{d}t^2 + \frac{\tilde{x}^2}{\alpha^2} \tilde{d}x^2 = -\tilde{d}t^2 + e^{\frac{x}{\tilde{w}}} dt^2 \quad \tilde{d}s^2 = -\tilde{d}t^2 + e^{2Ht} dt^2
\end{align*}
\]

where \( H \equiv \alpha^{-1} \). Now \(-\infty < \tilde{x} < 0 \), \(-\infty < \tilde{x} < \infty \), but covers only half of de Sitter.

In the full 4-chain de Sitter space we define \( \tilde{x} = \frac{\alpha x}{\tilde{w}} \), \( \tilde{y} = \frac{\alpha y}{\tilde{w}} \), \( \tilde{z} = \frac{\alpha z}{\tilde{w}} \)

and get the warping

\[
ds^2 = -\tilde{d}s^2 + \alpha(\tilde{w})^2 \left[ \tilde{d}x^2 + \tilde{d}y^2 + \tilde{d}z^2 \right]
\]

where \( \alpha(\tilde{w}) = e^{\tilde{x}/\tilde{w}} = e^{\tilde{H}t} \), where \( H = \frac{1}{\alpha} \frac{d\tilde{w}}{dt} = e^{-\frac{x}{\tilde{w}} - \frac{1}{\alpha}} \tilde{e}^{\tilde{x}/\tilde{w}} = \alpha^{-1} \isin t \).

\[
\therefore \text{an exponentially expanding flat universe.}
\]
de Sitter space and Einstein equation

de Sitter is a maximally symmetric 4-dim manifold, with $R > 0$

$$R_{00} = \kappa (g_{00}g_{00} - 3g_{00}g_{00}), \quad \kappa = \frac{R}{n(n-1)} = \frac{R}{12}$$

$$R_{00} = \kappa (n-1)g_{00} = 3\kappa g_{00} = \frac{1}{4} Rg_{00}$$

$$\therefore \quad G_{00} = R_{00} - \frac{1}{2} Rg_{00} = (\frac{1}{4} - \frac{1}{2}) Rg_{00} = -\frac{1}{4} Rg_{00} = -3\kappa g_{00}$$

Thus de Sitter is a solution of the Einstein eq. $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ if $T_{\mu\nu} = \frac{-3\kappa}{8\pi G} g_{\mu\nu}$

Interpreted as $T_{\mu\nu} = (\kappa + p) g_{\mu\nu} + p g_{\mu\nu}$ (the ideal fluid energy tensor), we see that

$$\kappa + p = 0 \quad \Rightarrow \quad p = -\kappa$$

And $p = -\frac{3\kappa}{8\pi G}$

$$\therefore \quad \kappa = \frac{-p}{\frac{3\kappa}{8\pi G}} = \text{const}$$

This is the energy tensor of vacuum energy, i.e., if vacuum ("empty space") has a non-zero energy density, it must be a constant, and Lorentz-invariant

$$\Rightarrow T_{\mu\nu} \propto g_{\mu\nu} \Rightarrow T_{\mu\nu} = -3\kappa g_{\mu\nu}, \quad \text{where} \quad \kappa = \text{const}$$

$$\therefore$$ de Sitter space describes a vacuum universe with constant positive energy density.
Poincare diagram for de Sitter space

The $t, \nu, \theta, \phi$ coords cover the entire spacetime

$$0 \leq \nu \leq \pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi$$

Here already finite range, but $-\infty < t < \infty$

$$ds^2 = -dt^2 + \frac{a^2 \cosh^2 \frac{t}{a}}{\sin^2 \nu} \left[ d\nu^2 + \sin^2 \nu \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]$$

Each circle is really a 3-sphere.

The figure is a 2-dim slice of de Sitter space. Depending on the orientation of $a$, the side circle would represent $\phi = 0 \to 2\pi$, or a half-circle $\theta = 0 \to \pi$

or $\nu = 0 \to \pi$

Choosing the last one: the half-circles make up half of the figure, where each point then represents a $(\theta, \phi)$ slice of the full de Sitter space.

To make this into a Poincare diagram, compactify $t$ by red transformation

A good choice is $t'$ where $\cosh \frac{t}{a} = \frac{t'}{a}$

$$\Rightarrow ... \Rightarrow dt'^2 = \frac{a^2}{\cos^2 t'} \frac{dx^2}{dx'^2}$$

$$\Rightarrow ds^2 = \frac{a^2}{\cos^2 t'} \left[ -dt'^2 + d\nu^2 + \sin^2 \nu d\Omega^2 \right]$$

$\Rightarrow$ light cones in $(\nu, t')$ come at $45^\circ$

\[ t' = \frac{\nu}{2} \]

\[ t' = -\frac{\nu}{2} \]

Observe at A and B cannot have met or been in communication in the past.

Observe at C and D cannot meet or communicate in the future.

Poincare diagram. Each point represent a 2-space, except those at the edges $\nu = 0, \pi$, which are points.
Anti-de Sitter space

The spacetime (Lorentzian signature) with constant negative curvature, $\text{AdS}^n$, is called anti-de Sitter space. It can be embedded in an $n+1$-dimensional flat spacetime, with metric (for $n=4$)

$$ds^2_{\text{AdS}} = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$$

as the surface

$$-u^2 - v^2 + x^2 + y^2 + z^2 = -\alpha^2$$

Study this in $n=2$ dimensions

$$ds^2_{(2)} = -du^2 - dv^2 + dx^2$$

$$-u^2 - v^2 + x^2 = -\alpha^2$$

Introduce new's $t'$ and $g$

$$u = \alpha \sinh t' \cosh g$$
$$v = \alpha \cosh t' \sinh g$$
$$x = \alpha \sinh g$$

$$\Rightarrow -u^2 - v^2 + x^2 = -\alpha^2 \left( \frac{\sinh^2 t' + \sinh^2 g}{1} \right)$$

Find the induced metric for $\text{AdS}^2$

$$du = \alpha \cosh t' \sinh g \ dt' + \alpha \sinh t' \cosh g \ dg$$
$$dv = -\alpha \sinh t' \sinh g \ dt' + \alpha \cosh t' \cosh g \ dg$$
$$dx = \alpha \cosh g \ dg$$

$$ds^2 = -du^2 - dv^2 + dx^2 = -\alpha^2 \left( \cosh^2 t' \sinh^2 g \ dt'^2 + 2 \cosh t' \sinh t' \sinh g \ dt' dg + \cosh^2 t' \sinh^2 g \ dg^2 \right)$$
$$-\alpha^2 \left( \cosh^2 t' \sinh^2 g \ dt'^2 - 2 \sinh t' \cosh t' \sinh g \ dt' \cosh g \ + \cosh^2 t' \sinh^2 g \ dg^2 \right) + \alpha^2 \cosh^2 g \ dg^2$$

$$= -\alpha^2 \sinh^2 g \ dt'^2 + \alpha^2 \left( \sinh^2 g - \sinh^2 t' \right) \ dg^2 = \alpha^2 \left( -\sinh^2 g \ dt'^2 + dg^2 \right)$$

We identify $t'$ as the timelike and $g$ as the spacelike coordinate.

The metric of $\text{AdS}^2$ is

$$ds^2 = \alpha^2 \left( -\sinh^2 g \ dt'^2 + dg^2 \right)$$  (1)
In the full $n=4$ case we introduce nd's $t', g, \theta, \phi$

$u = \cosh t' \cosh g$
$v = \cosh t' \sinh g$
$x = \sinh t' \cosh \theta$
$y = \sinh t' \sinh \theta \sin \phi$
$z = \sinh t' \sinh \theta \cos \phi$

Each pair of points $(t', g, \theta, \phi)$ and $(t' \pm 2\pi, g, \theta, \phi)$ in AdS$^2$ is replaced by

$a 2$-sphere $(\theta, \phi)$ of radius $|\sinh g|$

$ds^2 = \alpha^2 \left[ -\cosh^2 g dt'^2 + d\theta^2 + \sinh^2 g \left( d\phi^2 + \sin^2 \theta d\theta^2 \right) \right]$ (2)

This is the metric of $H^3$ (Exercise)

The time nd $t'$ is periodic, $t'$ and $t' + 2\pi$ give the same $(u, v, x, y, z)$, i.e., the same point on our surface embedded in flat 5-dim space.

However, when considering the 4-dim spacetime specified by the metric (2), there is no reason to require that $(t', g, \theta, \phi)$ and $(t' + 2\pi, g, \theta, \phi)$ are the same point. The periodicity is just an artifact of the way we defined the metric (2) from a particular embedding.

Therefore we consider instead the covering space of the surface discussed above: the spacetime with metric (2), but $t'$ extending from $-\infty$ to $+\infty$. This is the anti-de Sitter space.

For the $n=2$ de Sitter space we could have made the same argument about the nd $\phi$, allowing it to range from $-\infty$ to $+\infty$ representing different points on the manifold.

But for $n \geq 3$ we cannot do it anymore, because with additional dimensions we find the space "closed in" for $\theta \to 0$ ($\sin^2 \theta \to 0$). To avoid the manifold becoming irregular at $\theta=0$ we have to assume the space is periodic in $\phi$. 
· Other odd systems on AdS.

For an observer at constant $y, \theta, \phi$ the proper time is given by

$$\rho \, dt^2 = a^2 \sin^2 \rho \, dt'^2 \quad \Rightarrow \quad dt = a \sin^2 \rho \, dt'$$

Thus $t'$ does not measure this time.

· It is possible to replace $t', \rho$ with new coordinates $t, \alpha$ (exercise) so that the metric becomes

$$ds^2 = -dt^2 + a^2 \cos^2 \frac{t}{\alpha} \left[ d\alpha^2 + \sinh^2 \alpha \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right]$$

However, this odd system covers only a part of its spacetime, hitting odd singularities at $t = \pm \frac{\pi}{2} \alpha$. 
Penrose diagram for AdS

- The ends $t \in [0, \pi]$, $q \in [0, 2\pi)$ have a finite range, but $t' \in (-\infty, 0)$, $q \in (-\infty, 0)$ have an infinite range.

Try to bring the remaining ends to a finite range and make lightcone $45^\circ$.

It turns out that because of the 2nd requirement (45°) only the spatial coordinates can be unphysical. Thus is done by defining a new radial coordinate $g'$ by

$$\cosh g = \frac{1}{\cosh g'} \Rightarrow \sinh g \, dg = \frac{\sinh g'}{\cosh^2 g'} \, dg', \quad g' \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

$$\sinh^2 g = \cosh^2 g - 1 = \frac{1}{\cosh^2 g'} - 1 = \frac{\sinh^2 g'}{\cosh^2 g'}$$

$$\Rightarrow \, dg = \frac{1}{\sinh g} \cdot \frac{\sinh g'}{\cosh^2 g'} \, dg' = \frac{\sinh g'}{\cosh^2 g'} \cdot \frac{\sinh g'}{\cosh^2 g'} \, dg' = \frac{dg'}{\cosh g'}$$

$$\therefore \, ds^2 = g'^2 \left[ - \cosh^2 g \, dt^2 + dg^2 + \sinh^2 g \left( d\theta^2 + \sin^2 \theta \, d\psi^2 \right) \right]$$

$$= g'^2 \left[ - \frac{dt^2}{\cos^2 g'} + \frac{d\theta^2}{\cos^2 g'} + \frac{\sin^2 \theta}{\cos^2 g'} \left( d\psi^2 + \sin^2 \theta \, d\phi^2 \right) \right]$$

$$= g'^2 \left[ - \frac{dt^2}{\cos^2 g'} + \frac{d\theta^2}{\cos^2 g'} + \frac{\sin^2 \theta}{\cos^2 g'} \left( d\psi^2 + \sin^2 \theta \, d\phi^2 \right) \right]$$

metrical at 2-sphere

metrical at 3-sphere

defined that because the conformal factor depends also on $g'$, the $(\theta, \psi, \phi)$ does not have the unit of a 3-sphere, neither does it have the topology of a 3-sphere, since $g'$ ranges only from 0 to $\frac{\pi}{2}$, not to $\pi$.

(Negative value of $g'$: $g' = -g'$ gives the same point as $g + \pi - \theta$)

$\theta + \phi + \pi$

Each pair of points $(t', g') \& (t', -g')$ represents a 2-sphere, except points at $g' = 0$ represent a point (the horizon at our spherical coordinates). $g' = \pm \frac{\pi}{2}$ represents a sphere at spatial infinity.

The causal structure of AdS is very different from ds. Now every observer sees to spatial infinity. It is possible for any two observers to communicate and meet in the future (and have not in the past.)
8.2 Robertson-Walker Metric

Assume universe is spatially homogeneous and isotropic, but evolving in time

\[ M = \mathbb{R} \times \Sigma, \quad t \in \mathbb{R}, \quad \Sigma \text{ is a maximally symmetric 3-manifold} \]

(spatially \(\Leftrightarrow\) Euclidean signature)

\[ ds^2 = -dt^2 + a^2(t) \, dx^2 \]

where \(a^2 \) is the scale factor (another notation in \( R(t) \))

\[ ds^2 = g_{ij}(x^1, x^2, x^3) \, dx^i \, dx^j \]

This means that there exists such a coordinate system, which foliates the spacetime into homogenous and isotropic three slices. It is called \textit{cosmic time}.

These are called \textit{comoving} cd's. Only an observer at rest (const \( x^i \)) sees the universe as isotropic.

"Comoving observer"

From earlier discussion we know:

1. \( R_{ijkl} = k \left( y_{ik} y_{jl} - y_{il} y_{jk} \right) \)
2. \( R_{ij} = k (n-1) y_{ij} = 2k y_{ij} \)
3. \( R = k n (n-1) = 6k \) or \( k = \frac{6R}{n} \)

The maximally symmetric space is certainly spherically symmetric.

From our discussion of the Schwarzschild metric, we have that

\[ ds^2 = -A(r) \, dt^2 + B(r) \, dr^2 + r^2 \, d\Omega^2 = -e^{2\alpha(r)} \, dt^2 + e^{2\alpha(r)} \, dv^2 + r^2 \, d\Omega^2 \]

(Still spherically symmetric spacetime), we know that \( ds^2 \) can be put in form

\[ ds^2 = B(r) \, dv^2 + r^2 \, d\Omega^2 = e^{2\alpha(r)} \, dv^2 + r^2 \, d\Omega^2 \]

We can also borrow the calculation of the Schwarzschild \( R_{uv} \), by setting \( \alpha = 0 \)

\[ \begin{cases} 
R_{11} = \frac{2}{r^2} \beta^2 & = 2k y_{11} = 2k e^{2\alpha} \\
R_{22} = e^{2\alpha} (r \beta' - 1) + 1 & = 2k y_{22} = 2k r^2 \\
R_{12} = 0 \end{cases} \]

\[ e^{2\alpha} = \frac{1}{kr} \beta \]

\[ 1 - 2kr^2 = e^{2\alpha} (1 - r \beta) = \frac{kr}{r^2} (1 - r \beta) \]

\[ \frac{kr}{r^2} = 1 - kr^2 \Rightarrow \beta = \frac{kr}{1 - kr^2} \Rightarrow \beta = -\frac{1}{2} \ln (1 - kr^2) + C \]

Insert into (i) \( \Rightarrow \frac{e^{2\alpha}}{1 - kr^2} = \frac{1}{1 - kr^2} \Rightarrow C = 0 \)

\[ e^{2\alpha} \beta = \frac{1}{1 - kr^2} \]
Thus we have \[ ds^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \] where \[ k = \frac{(3)R}{6} = \text{const} \]

The full spacetime has metric

\[ ds^2 = -dt^2 + \alpha^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right] \tag{1} \]

We are free to rescale the radial coordinate \( r \) by replacing

\[ \tilde{r} = \lambda r, \quad \tilde{\alpha} = \frac{1}{\lambda} \alpha, \quad \tilde{k} = \frac{1}{\lambda^2} k \]

(This rescaling changes the size of the manifold \( \Sigma \), and thus the corresponding \((3)R\), as its size is related to physical size by the scale factor \( \alpha \)).

There are two common ways to rescale:

1) Unless \( k = 0 \), rescale to \( k = \pm 1 \)

2) Rescale so that \( \alpha(t_x) = 1 \) at some reference time \( t_x \),
   e.g. the present time \( t_0 \)

We have three cases, depending on the sign of \((3)R\) (or, equivalently \( k \))

\((3)R = 0\): No curvature on the 3-space \( \Sigma \), we say the universe is flat.

\((3)R > 0\): \( \Sigma \) has positive curvature, it has the geometry of \( S^3 \),
   we say the universe is closed.

\((3)R < 0\): \( \Sigma \) has negative curvature, it has the geometry of \( H^3 \),
   we say the universe is open.

We have not yet used Einstein's equation. It will determine \( \alpha(t) \).
Geometry of space

$$ds^2 = -dt^2 + a(t)^2 dx^2$$

- **k = 0**
  $$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 = dx^2 + dy^2 + dz^2$$

  We say the universe is "flat".

- **k = +1**
  $$ds^2 = \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

  Coordinate singularity at \( r = 1 \) \((1/r > 1)\) would create a "big bang" do not would that.

  Define new radial coordinate \( \xi \) by \( r = \sin \xi \)

  $$ds^2 = d\xi^2 + \sin^2 \xi (d\theta^2 + \sin^2 \theta d\phi^2)$$

  Models at 3-sphere \( S^3 \) of radius \( 1 + a(t) \)

  We say the universe is "closed"

  circumference \( 2\pi a \), volume \( 2\pi^2 a^3 \)

- **k = -1**
  $$ds^2 = \frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

  No coordinate singularity. We can anyway define \( r = \sinh \xi \)

  $$ds^2 = d\xi^2 + \sinh^2 \xi (d\theta^2 + \sin^2 \theta d\phi^2)$$

  Models at 3-hyperboloid \( H^3 \)

  a space of constant negative curvature.

  \( r \) (and \( \xi \)) range from 0 to \( \infty \)

  We say the universe is "open".

  - For \( k = +1 \), the universe is finite.
  - For \( k = 0, -1 \), the universe is infinite. Or at least that is the simplest assumption.

It is possible that the universe has a nontrivial topology, so that it is finite even in these cases.
Conformal Time

- It is often useful to make a change in the time coordinate from the cosmic time $t$ to the conformal time $\eta$, defined by

$$d\eta = \frac{dt}{a(t)}$$

or

$$\eta = \int_0^t \frac{dt'}{a(t')}$$

Sometimes some other lower limit (must be) used.

The RW metric becomes

$$ds^2 = a(\eta)^2 \left[ -d\eta^2 + \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

- For $k=0$, we have

$$ds^2 = a(\eta)^2 \left[ -d\eta^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] = a(\eta)^2 \left[ -d\eta^2 + dx^2 + dy^2 + dz^2 \right]$$

- For $k = \pm 1$, we can use $x$ as the radial coordinate:

$$ds^2 = a(\eta)^2 \left[ -d\eta^2 + dx^2 + \left( \frac{\sin^2 \theta}{\sin^2 \theta} \right) (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

This form is especially nice for studying radial light propagation, because the light-like condition becomes $dy = dx$ (for $k=0$, already had $dy = dr$)

In the end one may want to convert back to cosmic time $t$ to interpret the results.

*) Sometimes the conformal time is defined

$$d\eta = \frac{a(t_x)}{a(t)} dt$$

where $t_x$ is some reference time (e.g., the present time).
8.3 Friedmann Equation

Calculating the Ricci tensor for the Robertson-Walker metric
\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \]
gives (exercise) the $R_{\mu\nu}$ at the bottom of this page and \( \rho \equiv \frac{\dot{a}}{a} \)

\[ R_{00} = -3 \frac{\ddot{a}}{a} \quad R_{0\nu} = 0 \quad \text{for } \mu \neq \nu \]

\[ R_{11} = \frac{a\dddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \]

\[ R_{22} = r^2(a\dddot{a} + 2\dot{a}^2 + 2k) \]

\[ R_{33} = r^2(a\dddot{a} + 2\dot{a}^2 + 2k) \sin^2\theta \]

and
\[ R = g^{\mu\nu} R_{\mu\nu} = 6 \left[ \frac{\dddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \]

Assume the energy-momentum tensor has the perfect fluid form
\[ T_{\mu\nu} = (\rho+p)u_\mu u_\nu + pg_{\mu\nu} \]

The assumption of isotropy requires that the fluid is at rest in the frame whose the metric is isochoric. Thus

\[ u^\mu = (1,0,0,0) \quad \Rightarrow \quad u_\mu = (-1,0,0,0) \]

\[ T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \rho \end{pmatrix} \quad \Rightarrow \quad T_{\mu\nu} = \text{diag}(-\rho, \rho, \rho, \rho) \]

\[ T = T_{\mu\nu} = -\rho + 3p \]

<table>
<thead>
<tr>
<th>$\Gamma^0_{11}$</th>
<th>$\frac{a\ddot{a}}{1-kr^2}$</th>
<th>$\Gamma^0_{22} = a\dot{a}r^2$</th>
<th>$\Gamma^0_{33} = a\dot{a}r^2 \sin^2 \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma^1_{11}$</td>
<td>$\frac{kr}{1-kr^2}$</td>
<td>$\Gamma^1_{22} = -r(1-kr^2)$</td>
<td>$\Gamma^1_{33} = -r(1-kr^2) \sin^2 \theta$</td>
</tr>
<tr>
<td>$\Gamma^2_{12}$</td>
<td>$\frac{\dot{a}}{a}$</td>
<td>$\Gamma^2_{12} = \frac{1}{r}$</td>
<td>$\Gamma^2_{33} = -\sin \theta \cos \theta$</td>
</tr>
<tr>
<td>$\Gamma^3_{13}$</td>
<td>$\frac{\dot{a}}{a}$</td>
<td>$\Gamma^3_{13} = \frac{1}{r}$</td>
<td>$\Gamma^3_{33} = \cot \theta$</td>
</tr>
</tbody>
</table>
The Einstein eqn. can be written

\[ R_{\mu\nu} = 8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) \] or \[ R_{\mu\nu}^\nu = 8\pi G (T_{\mu\nu}^\nu - \frac{1}{2} g_{\mu\nu}^\nu T) \]

\[ T_0^0 - \frac{1}{2} T = -\frac{2}{3} - \frac{1}{2}(\varphi + \frac{3}{2}p) = -\frac{1}{2} - \frac{1}{2}p \]

\[ T_1^1 - \frac{1}{2} T = p - \frac{1}{2}(\varphi + \frac{3}{2}p) = +\frac{1}{2} - \frac{1}{2}p \]

\[ R_2^2 = R_3^3 = R_1^1 \]

"The 2nd Friedmann equation"

- The \[ \mu = \nu = 0 \] eqn. gives

\[ \ddot{a} - \frac{\dot{a}}{a} = -\frac{4\pi G}{3}(\varphi + 3p) \] (1)

- \[ \mu = \nu = 1, 2, 3 \] give

\[ a \ddot{a} + 2a \dot{a}^2 = 4\pi G (\varphi - p) a^2 \] (2)

- \[ \mu \neq \nu \] give

\[ 0 = 0 \]

Using (1) to eliminate \[ \ddot{a} \] from (2) gives the Friedmann equation

\[ (\dot{a}/a)^2 = \frac{8\pi G}{3} \varphi - \frac{k}{a^2} \] (3)

\[ H(a) = \frac{\dot{a}}{a} \] Hubble parameter

- The energy-momentum continuity equation \[ T_{\mu\nu}^\nu = 0 \] follows from the

Einstein equation (due to the Bianchi identity). For a perfect fluid it becomes

due to the two equations (see p. 1-14)

\[ (g_{\mu\nu})_{\mu}^\nu + p g_{\mu\nu} = 0 \]

\[ (g + p) u_{\mu}^\nu u_{\mu}^\nu = -(g_{\mu\nu} + u_{\mu}^\nu u_{\mu}^\nu) p_{\mu} \]

The first one can be written

\[ g_{\mu\nu} u_{\mu}^\nu + (g + p) u_{\mu}^\nu u_{\mu}^\nu = g_{\mu\nu} u_{\mu}^\nu + (g + p) u_{\mu}^\nu u_{\mu}^\nu + (g + p) \Pi_{\mu\nu} u_{\mu}^\nu = 0 \]

With \[ u_{\mu}^\nu = 0 \] this becomes

\[ g_{\mu\nu} + (g + p) \Pi_{\mu\nu} = 0 \] (4)

This energy continuity equation can also be derived (exercise) from Eqs (1) and (3).

Thus the 3 eqns. (1), (3), and (4) for the 3 quantities \( \dot{a}, p(t), g(t) \)

are not all independent. To be able to solve them, we still need an

equation of state: \( p(g) \), relating \( p \) and \( g \).
A particularly simple equation of state is

\[ p = w g \quad \text{with} \quad w = \text{const} \]

In practice, there are three such cases:

1. \( w = 0 \) "dust" or "matter" (modeled at velocities whose thermal velocities are nonrelativistic, i.e., \( v \ll c \Rightarrow p \ll g \))

2. \( w = \frac{1}{3} \) "radiation" (modeled at very low velocities, like photons, or ultrarelativistic particles, whose thermal velocities are \( \sim c \)).

3. \( w = -1 \) "vacuum energy"

In general, we can define an equation of state parameter \( w \equiv \frac{p}{g} \)
(We are too assuming a homogeneous universe, so \( \frac{p(t)}{g(t)} = w(t) \).

The energy continuity eq becomes

\[ \frac{\dot{g}}{g} = -3(1+w) \frac{\dot{a}}{a} \]

- If \( w = \text{const} \), \( g \propto a^{-3(1+w)} \)

  \begin{align*}
  \text{Matter:} & \quad g \propto a^3, \quad 8a^3 = \text{const}, \quad g = g_0 \left( \frac{a}{a_0} \right)^3 \\
  \text{Radiation:} & \quad g \propto a^4, \quad 8a^4 = \text{const}, \quad g = g_0 \left( \frac{a}{a_0} \right)^4 \\
  \text{Vacuum energy:} & \quad g = \text{const} \\
  \end{align*}

- If all three components are present,

\[ g = g_0 \left( \frac{a}{a_0} \right)^4 + g_{\text{rad}} \left( \frac{a}{a_0} \right)^3 + g_{\text{vac}} \]
The Einstein Static Universe (also called just the "Einstein Universe")

- Look for a static solution $a, \theta, \rho = \text{const} \Rightarrow \dot{a} = \dot{\theta} = \dot{\rho} = 0$

The energy continuity equation (4) is automatically satisfied:

(1) $\Rightarrow \rho = -\frac{1}{3} \dot{\theta}$

(2) $\Rightarrow k = \frac{8\pi G}{3} \dot{a}^2$

Assuming $\theta > 0$, this leads to negative pressure and positive curvature.

(3) $R = 6k = 16\pi G \rho \dot{a}^2$ for $k = 3$-metric $d\theta^2$ (i.e. $w/o a^2$)

$R = 6\left[ \frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2} \right] = \frac{6k}{a^2} = 16\pi G \theta$

One gets $\rho = -\frac{1}{3} \dot{\theta}$ by the combination $\theta = 3m + 3\text{vac}$

where $3m$ represents ordinary matter with $\rho_m = 0$

and $3\text{vac}$ represents vacuum energy with $\rho_{\text{vac}} = -\frac{1}{3} \dot{\theta}$

when $3\text{vac} = \frac{1}{2} 3m$.

∴ $\rho = \rho_{\text{vac}} = -\frac{1}{3} \dot{\theta}$

$\theta = 3m + 3\text{vac} = \frac{3}{2} 3m$.

The Einstein Universe requires this perfect balance between matter and vacuum energy. It is unstable to perturbations.

We now know that the universe is expanding, not static.

However, the Einstein universe still has theoretical importance (related to Penrose diagrams).

Its geometry is that of $R \times S^3$. Its metric is

$$\text{d}s^2 = -\text{d}t^2 + a^2 \left[ \frac{\text{d}r^2}{1-kr^2} + r^2(\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2) \right]$$

$$= -\text{d}t^2 + a^2 \left[ \text{d}r^2 + \sin^2 \theta \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \right) \right]$$

$\text{d}\Omega_{(3)}^2$

$\in (-\infty, \infty)$

$\theta \in [0, \pi]$ 

$\phi \in [0, 2\pi)$

where $a = \text{const.}$
8.4 Evolution of the Scale Factor

- The spacelike with RW metric, the scale factor $a(t)$ solved from the
Friedmann eq, are called Friedmann-Robertson-Walker universes.

\[ \text{FRW} (-1) : \text{the open, } k < 0, \text{ case} \quad \text{(for } w = 0, \text{ these are} \]
\[ \text{FRW} (0) : \text{the flat, } k = 0, \text{ case} \quad \text{called the} \]
\[ \text{FRW} (1) : \text{the closed, } k > 0, \text{ case} \]

- For simplicity, we consider only the flat case, FRW (0),
and only for the case $p = w q$, $w = \text{const.}$. For $w = 0$, this is called the
Einstein-de Sitter universe.

\[ ds^2 = -dt^2 + a(t)^2 \left[ dr^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2 \right] \]
\[ = -dt^2 + a(t)^2 \left[ dx^2 + dy^2 + dz^2 \right] \quad (1) \]

The Friedmann eq is now

\[ \left( \frac{a}{a_0} \right)^2 = \frac{8 \pi G}{3} \frac{\rho}{\rho_0} = \frac{8 \pi G}{3} \frac{\rho}{\rho_0} \left( \frac{a}{a_0} \right)^{-3(1+w)} \quad (2) \]

- Solve now the Friedmann eq. for $a(t)$

\[ (1) \Rightarrow \frac{a}{a_0} = \sqrt{\frac{8 \pi G}{3} \frac{\rho}{\rho_0} \left( \frac{a}{a_0} \right)^{-3(1+w)}} \]
\[ \Rightarrow a^{3(1+w)-1} da = \sqrt{\frac{8 \pi G}{3} \frac{\rho}{\rho_0} \frac{3}{a_0} (1+w)} \quad (3) \]

- For $w > -1$ we integrate (3) to

\[ \frac{2}{3(1+w)} a^{3(1+w)} = \sqrt{\frac{8 \pi G}{3} \frac{\rho}{\rho_0} \frac{3}{a_0} (1+w)} \left( t - t_c \right) \quad (4) \]

\[ \Rightarrow a = a_0 \left[ \frac{3(1+w)}{2} \sqrt{\frac{8 \pi G}{3} \frac{\rho}{\rho_0}} \right]^{\frac{2}{3(1+w)}} \left( t - t_c \right)^{\frac{2}{3(1+w)}} \]

\[ \therefore a(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow t_c \quad \text{singularity!} \quad \text{(for } w > -1) \]

This is a true singularity, where $\rho \rightarrow \infty$. The spacetime ends there (in the
past direction), and we choose this as the origin of our time coordinate,
i.e., we choose $t_c = 0$. 

\[ \therefore a(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow t_c \quad \text{singularity!} \quad \text{(for } w > -1) \]
We get \( a \propto t^{\frac{2}{3(1+w)}} \) for \( w = -1 \).

For matter, \( a \propto t^{\frac{2}{3}} \). For radiation, \( a \propto t^{\frac{1}{2}} \).

We can write (4) as

\[
\frac{a}{a_o} = \left( \frac{t}{t_o} \right)^{\frac{2}{3(1+w)}} \quad (5)
\]

where \( t = t_o \) is the present time, when \( a = a_o \).

The universe keeps expanding forever. It has a finite past, an infinite future.

For \( w = -1 \) (vacuum energy), (5) reads

\[
\frac{\text{d} a}{a} = \sqrt{\frac{8\pi G}{3}} \frac{\text{d} t}{t_0} \quad \Rightarrow \quad \ln \frac{a}{a_o} = \sqrt{\frac{8\pi G}{3}} \frac{t-t_0}{t_o} \quad (6)
\]

The universe expands exponentially. There is no singularity. The universe has an infinite past, and an infinite future.

As we discussed on p. 8-6 this is the de Sitter space. (Or half of it, i.e., the \( t,x,y,z \) cells cover half of the de Sitter space; but the space-time can be maximally extended (introducing another coordinate system) to fill the de Sitter space.)
From (5), \( a \propto t^q \), where \( q = \frac{2}{3(1+w)} \) for \( w = \text{const} \).

For \( w = -\frac{1}{3} \): \( q = 1 \) and \( a \propto t \), \( \ddot{a} = \text{const} \), \( \dddot{a} = 0 \)

\( w > -\frac{1}{3} \): \( q < 1 \), the expansion is decelerating

\( w < -\frac{1}{3} \): \( q > 1 \), the expansion is accelerating

**Acceleration and Deceleration**

We do not have realistic models for cosmology where \( w = \text{const} \) for other values of \( w \) down to \(-1, 0, \frac{1}{3}\), but the above conclusions hold for non-constant \( w \):

From the "2nd Friedmann equation" \( \frac{\dddot{a}}{a} = -\frac{4\pi G}{3} (1+3w) \rho \),

we see (assuming \( \rho > 0 \)) that the expansion is

accelerating \((\dddot{a} > 0)\) for \( w < -\frac{1}{3} \),

decelerating \((\dddot{a} < 0)\) \( w > -\frac{1}{3} \).

This holds for all values of \( k \).

In a universe containing the three components \( \rho = \rho_r + \rho_m + \rho_v \),

\[
W = \frac{\rho_r + \rho_m + \rho_v}{\rho_r + \rho_m + \rho_v} = \frac{5\rho_r - 8\rho_v}{5\rho_r + 8\rho_m + 8\rho_v} = \frac{5}{8} W_0 = \frac{1}{3} \frac{\rho_r}{\rho} - \frac{8\rho_v}{\rho}
\]
denotes as the universe expands. Since \( \rho_r \propto a^{-4} \), \( \rho_m \propto a^{-3} \), \( \rho_v = \text{const} \),
the universe evolves from radiation domination \((w = 1/3)\), through matter domination \((w = 0)\) to vacuum domination \((w = -1)\), if the universe expands sufficiently. Thus at some point deceleration turns into acceleration.

According to observations, the expansion of our universe is now accelerating. Whether this is due to a positive vacuum energy, some other unknown energy component with negative pressure ("dark energy"), or a required modification of GR, is not known.
Penrose Diagram for FRW(0) (decelerating case)

- For simplicity, consider the case \( w = 0 \). The Penrose diagram looks the same also for other FRW(0) universes, as long as \( w > -\frac{1}{3} \) (expansion in deceleration).

The metric is

\[
adv^2 = -dt^2 + \left(\frac{t}{t_0}\right)^{4/3} (dr^2 + r^2 d\Omega^2) \quad (a_0 = 1)
\]

- Introduce conformal time \( \tau \) by

\[
dt = a(t) dy = \left(\frac{t}{t_0}\right)^{2/3} dy \quad \Rightarrow \quad dy = t_0^{-2/3} t^{2/3} dt
\]

\[
\Rightarrow \quad \tau = 3 t_0^{2/3} t^{1/3} = 3 t_0 \left(\frac{t}{t_0}\right)^{1/3}
\]

( \( \Rightarrow \tau_0 = 3 t_0 \), the "conformal age" of the universe in 3x actual age )

The metric becomes

\[
adv^2 = \left(\frac{t}{t_0}\right)^{4/3} (-dt^2 + dr^2 + r^2 d\Omega^2)
\]

\[
= \left(\frac{\tau}{\tau_0}\right)^4 (-d\tau^2 + dr^2 + r^2 d\Omega^2)
\]

conformal factor

It is conformal to the Minkowski metric.

- The coordinates \((\eta, r)\) have the lightcone 45° property, but they are not compact.

\(0 < \eta < \infty, \quad 0 \leq r < \infty\)

1) \( u = \eta - r \)

\( v = \eta + r \)

2) \( U = \text{arctan} \, u \)

\( V = \text{arctan} \, v \)

3) \( T = V + U \)

\( R = V - U \)

To compactify them while retaining the 45° property, we do as we did for the Schwarzschild metric: turn to null coordinates for compactification.
\[ ds^2 = \frac{1}{\omega(T,R)^2} \left( -dT^2 + dR^2 + \sin^2 R \, d\Omega^2 \right) \]

where the conformal factor \( \omega(T,R) \) is

\[ \omega(T,R) = A \left( \frac{\cos T + \cos R}{2 \sin T} \right)^{\frac{1}{2}} \left( \cos T + \cos R \right) \]

(Here \( A \) is a constant \( \times \frac{e^t}{t} \), which makes the dimension of \( ds^2 \) length-squared.
I did not bother to work out the value of this constant. Also, I should have defined \( u = \frac{y-R}{\eta_0}, \quad v = \frac{y+R}{\eta_0} \) to make \( u,v \) dimensionless.)

We are now not so interested in the conformal factor \( \omega(T,R) \). What is important, is that the cell's \( T,R \) have the \( 45^\circ \) property and they now have a great range.

What is this range? We have

\[ T = \text{arctan} (y+v) + \text{arctan} (y-v) \]
\[ R = \text{arctan} (y+v) - \text{arctan} (y-v) \]

The \( y=0 \) and \( r=0 \) cell lines become

\[ \begin{align*}
&\left\{ \begin{array}{l}
T = \text{arctan} (r) + \text{arctan} (-r) = 0 \\
R = \frac{\text{arctan} (r) - \text{arctan} (-r)}{2 \text{arctan} r}
\end{array} \right. \\
&\text{which ranges from 0 to } \pi \text{ for } r \in (0,\infty)
\]

and

\[ \begin{align*}
&T = \text{arctan} y + \text{arctan} y = 2 \text{arctan} y \\
&R = \text{arctan} y - \text{arctan} y = 0
\end{align*} \quad \text{for } y \in (0,\infty)
\]

The null infinity \( V=\infty \) becomes

\[ V = \frac{1}{2} (T+R) = \text{arctan} \infty = \frac{\pi}{2} \]

.: \( T > 0 \)
\( R > 0 \)
\( T+R < \pi \)

(this keeps \( \cos T + \cos R > 0 \))

\[ \eta = \text{const} \quad (or \, t=\text{const}) \]

singularity at \( t=0 \)

Any two observers have the possibility to meet in its future.
The past light cone at event covers only part of its universe.

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Fig. The Penrose diagram for FRW(0) with decelerating expansion.