1. Time dilation. Time dilation has been verified experimentally by flying accurate atomic clocks around the world. Consider a flight along the equator at altitude 10 km and velocity 300 m/s. How much does the time of the clock that travelled around the world differ from the time of a clock that remained still with respect to the Earth (at sea level)

(a) for an eastbound flight
(b) for a westbound flight.

Note that the Earth rotates. How much of the time dilation is due to gravity?

2. Radar time delay.

a) In a spacetime with Schwarzschild metric, a radar signal is sent from \((r_2, \theta_0, \phi_0)\) to \((r_1, \theta_0, \phi_0)\). The signal is immediately reflected and travels back again. Assume \(r_2 > r_1 > r_S\). The signal is radial (\(\theta = \theta_0, \phi = \phi_0\) stay constant) and lightlike (\(ds^2 = 0\)). Find the round-trip time \(\Delta \tau\) measured by an observer at \((r_2, \theta_0, \phi_0)\).

b) Since the distance between the two points is (you don’t have to calculate this)

\[
\Delta R = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{r_S}{r}}} = \sqrt{r_2(r_2 - r_S)} - \sqrt{r_1(r_1 - r_S)} + r_S \ln \left( \frac{\sqrt{r_2} + \sqrt{r_2 - r_S}}{\sqrt{r_1} + \sqrt{r_1 - r_S}} \right),
\]

one might naively expect the round-trip time to be

\[
\Delta \hat{\tau} = \frac{2\Delta R}{c} = 2\Delta R.
\]

The difference \(\Delta \tau - \Delta \hat{\tau}\) is called the time delay. Show that for \(r_1 \gg r_S\), the time delay is

\[
\Delta \tau - \Delta \hat{\tau} \approx r_S \left( \ln \frac{r_2}{r_1} - \frac{r_2 - r_1}{r_2} \right).
\]

Explain the cause of the time delay.

3. Schwarzschild orbits. From the Schwarzschild line element we get the equation

\[
\left(1 - \frac{r_S}{r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{r_S}{r}} - r^2 \dot{\phi}^2 = \beta
\]

where \(\beta = 1\) for massive particles and \(\beta = 0\) for massless particles like photons.

a) Using the constants of motion \(h\) and \(k\) for geodesic motion, convert this equation into the form

\[
\frac{1}{2} \dot{r}^2 + V(r) = \frac{1}{2} k^2,
\]

where \(V(r)\) depends on \(r_S\) and \(h\).

b) Draw \(V(r)\) (a typical case) for massive and massless particles.

c) The above equation has the same form as that of a particle moving in a 1-dimensional potential \(V(r)\), with total energy \(\frac{1}{2} k^2\). Using this analogy, explain how different values of \(k\) correspond to bound (motion limited to \(r_1 \leq r \leq r_2\)) and unbound (no upper limit to \(r\)) orbits, and orbits which lead to the black hole at the center. (You don’t have to solve the corresponding \(k\) values.)

d) Are there bound orbits for photons?

e) Explain how to locate stable and unstable circular \((r =\text{const.})\) orbits on the plot of \(V(r)\).

f) Are there stable/unstable orbits for massive particles? What about photons?

Please rotate the paper by \(\pi\) around the vertical axis.
4. **Dropping a beacon into a black hole.** Consider an observer sitting at constant spatial coordinates $(r_0, \theta_0, \phi_0)$ outside a Schwarzschild black hole of mass $M$, with $r_0 > r_S$. The observer drops a beacon into the black hole (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength $\lambda_{\text{em}}$ (in the beacon rest frame).

(a) Calculate the coordinate speed $dr/dt$ of the beacon, as a function of $r$.

(b) Calculate the beacon three-velocity measured by another observer sitting still at fixed $r$ as the beacon passes by. What is it at $r = r_S$?

(c) Calculate the wavelength $\lambda_{\text{obs}}$, measured by the observer at $r_0$, as a function of the radius $r_{\text{em}}$ at which the radiation was emitted.

(d) Calculate the time $t_{\text{obs}}$ at which a beam emitted by the beacon at radius $r_{\text{em}}$ will be observed at $r_0$.

(e) Show that at late times, the redshift grows exponentially: $\lambda_{\text{obs}}/\lambda_{\text{em}} \propto e^{t_{\text{obs}}/T}$. Give an expression for the time constant $T$ in terms of the black hole mass $M$. 