Due on Tuesday April 10 by 12.15.

1. **Gauge transformation of gravitoelectric and gravitomagnetic fields.**
   a) How do the gravitoelectric and gravitomagnetic fields $\vec{G}$ and $\vec{H}$ transform in a gauge transformation?
   b) There is a subclass of gauge transformations that leave $\vec{G}$ and $\vec{H}$ invariant. What is the condition for such gauge transformations?

2. **Test particle energy in linearized gravity.** We got from the geodesic equation the result
   \[
   \frac{dE}{dt} = -E \left[ \partial_t \Phi + 2\vec{v} \cdot \nabla \Phi - \frac{1}{2} (\partial_i w_j + \partial_j w_i - \partial_t h_{ij} ) v^i v^j \right],
   \]
   where $E = p^0$ is the time component of the test particle 4-momentum. This corresponds to the energy as observed by a fictitious 'background observer', who is still in the background coordinates and whose proper time is the background time $t$. Let's instead consider a physical observer, with four-velocity $u^\alpha = (u^0, u^i)$. Take the observer to remain at constant coordinates, i.e. $u_i = 0$. Find the equation for the time evolution of the observed energy $E_{\text{obs}} = -u_\alpha p^\alpha$ for a freely falling test particle, i.e.
   \[
   \frac{dE_{\text{obs}}}{dt} = \ldots
   \]
   Take $\vec{w} = 0$ (a possible gauge choice).

3. **Gravitomagnetic field of a rotating body.** Consider a homogeneous spherical shell with mass $M$, radius $R$, that is slowly rotating (i.e. calculate only to first order in velocity of mass motion) with angular velocity $\Omega$.
   a) Using the transverse gauge, find the gravitoelectric and gravitomagnetic fields $\vec{G}$ and $\vec{H}$ at large distances ($r \gg R$) caused by this rotating mass. (Note that this is stationary situation, so time derivatives of the metric vanish.)
   b) Sketch the $\vec{H}$ field lines.
   c) What will happen to a test particle that is dropped from a large distance? (Consider a particle at the equatorial plane; no calculations are required for this last question.)

4. **Effect of vector perturbations on geodesic motion.** Consider a metric where the scalar and tensor perturbations are zero, but not the vector perturbation. Take the vector perturbation to be constant in time and aligned with the $z$-axis, $w_i = f(x^k) \delta_{iz}$. Find the motion of a test particle that is not assumed to be initially at rest. (Assume that the test particle is moving at non-relativistic velocities, so that you can neglect terms that are non-linear in the velocity, but not crossterms between the velocity and the metric perturbation.)