Due on Tuesday May 2 by 12.15. These are the last exercises.

1. **Killing vectors.** Show that if $V$ and $U$ are Killing vectors, then $[V, U]$ is also a Killing vector.

2. **Deriving the FRW metric.** Let’s assume that hypersurfaces orthogonal to a timelike vector $u$ are maximally symmetric. Choose the time coordinate (labeled with 0) to be such that $u$ is the tangent vector of lines of constant time.
   a) Show that $g_{00} = 0$.
   b) Write the metric as $ds^2 = -dt^2 + a(t)^2(e^{2\alpha(r)} dr^2 + r^2 d\Omega^2)$, derive the Ricci scalar of the spatial part and use the fact that it is equal to the constant $6K$ to solve for $\alpha(r)$.

3. **No static universe.** Consider a FRW universe filled with matter, with $\rho > 0, p \geq 0$ and $\omega \equiv \rho/p = \text{constant}$.
   a) Show that there are no static solutions (i.e. solutions for which $a(t)$ is constant) for $\Lambda \leq 0$.
   b) Show that there is a static solution for $\Lambda > 0$, and that it is unstable.

4. **Age of the Universe.** The current standard model of cosmology is the $\Lambda$CDM model, where the universe is spatially flat ($k = 0$), and there are two main energy components, matter (cold dark matter (CDM) + ordinary matter) with $w = 0$ and vacuum energy ($\Lambda$) with $w = -1$. (Radiation is only important for the first few million years, so we ignore it here.) This model is a good fit to the Planck 2015 CMB data if the Hubble constant (the current value of the Hubble parameter $H(t)$) is $H_0 = 68 \text{ km/s/Mpc}$ and the total energy density is divided into 31% matter and 69% vacuum energy at present, $\Omega_{m0} = \rho_m(t_0)/\rho(t_0) = 0.31$ and $\Omega_{\Lambda 0} = \rho_{\text{vac}}(t_0)/\rho(t_0) = 0.69$.
   a) Find the age of the universe $t_0$.
   b) At what time $t_{\text{eq}}$ were the matter and vacuum energy densities equal?
   c) At the present time the expansion is accelerating, $\ddot{a} > 0$. When did the acceleration begin ($\ddot{a} = 0$)?
   (Hint: Use the substitution $x^{3/2} = b \sinh \phi$ in the integral $\int \frac{x^{1/2} dx}{\sqrt{b^2 + x^3}}$.)