5. COSMIC SHEAR  [SKW Part III, Sec. 6]

§5.1 Light Propagation in an Inhomogeneous Universe

The metric of an FRW universe with scalar perturbations, approximating the Barden potential to be equal, $\phi = \psi$ (good at least in the $\Lambda$CDM model during matter domination), can be written

$$ds^2 = a(t)^2 \left\{ - (1 + 2\psi) dt^2 + (1 - 2\psi) \left[ dw^2 + f_k(w) \left( ds^2 + \sin^2 \theta d\phi^2 \right) \right] \right\}$$  \hspace{1cm} (1)

where the Barden potential $\phi$ is a function of space and time.

Here $\tau$ denotes conformal time and $w$ is the radial comoving distance (in the unperturbed FRW universe).

Our observations lie on our past light cone $w(\tau) = \tau - \tau_0$.

Thus we need $\phi$ only on this light cone: $\tau(w) = \tau_0 - w$

$$\phi(x, w) = \phi(\tau_0 - w, x, w)$$

$\xi$ 2D coordinate vector in the direction transverse to the line of sight.
Consider two light rays arriving at the observer:
- the fiducial light ray corresponding to $\tilde{\theta} = 0$
- another ray observed at an angle $\tilde{\theta}$ from the fiducial ray

Without lensing ($\phi = 0$), the two light rays would be separated by $\tilde{x}(w) = f_k(w) \tilde{\theta}$

From homework 10.2, the deflection angle accumulates as $d\tilde{\alpha} = 2\nabla_\perp \phi \, dw$

In the homework both $\nabla_\perp$ and $\tilde{\theta}$ were in local coordinates; we can replace them with comoving coordinates by scaling both with $1+x$; hence cancel

$\Rightarrow \quad d\tilde{\alpha} = 2\nabla_\perp \phi \, dw \quad (2)$ where $\nabla_\perp$ is now written in comoving transverse coordinates $\tilde{x} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$

Suppose we have a finite number (in the figure below, two, at $w_1$ and $w_2$) of thin (thickness $\Delta w$) along the way:

$\tilde{x}(\tilde{\theta},w) = f_k(w) \tilde{\theta} - \sum_i \tilde{\alpha}_i f_k(w-w_i)$

$= f_k(w) \tilde{\theta} - 2 \sum_i f_k(w-w_i) \int_{\Delta w_i} \nabla_\perp \phi (\tilde{x}(\tilde{\theta},w),w') \, dw'$

$\Rightarrow f_k(w) \tilde{\theta} - 2 \int_{w}^{w'} dw' f_k(w-w') \nabla_\perp \phi (\tilde{x}(\tilde{\theta},w'),w')$

When we consider the entire distance from $0$ to $w$ as consisting of adjacent thin lenses, i.e., the light ray is being lensed all the way by the local $\nabla_\perp \phi$. 
The preceding ignored the bending of the fiducial (κ = 0) ray. For $\xi(\hat{\xi}, w)$ to represent the (moving) separation between these two rays, subtract the effect on the fiducial ray

$$\Rightarrow \xi(\hat{\xi}, w) = f_k(\hat{\xi})\hat{\xi} - 2\int_0^w f_k(w-w') [\nabla_\hat{\xi} \phi(\xi(\hat{\xi}, w'), w') - \nabla_\hat{\xi} \phi(\xi(\hat{\xi}, 0), w')]$$

(3)

\[ \phi \]

If there is a source at moving distance $w$ and we see at an angle $\hat{\xi}$ then its angular location on the same plane $\hat{\beta} = \frac{\xi(\hat{\xi}, w)}{f_k(\hat{\xi})}$

\[ \phi \]

The Jacobian for the total lensing effect from observer to moving distance $w$ in

$$A(\hat{\xi}, w) = \frac{\partial \hat{\beta}}{\partial \hat{\xi}} = \frac{1}{f_k(\hat{\xi})} \frac{\partial \xi}{\partial \hat{\xi}} = \Delta \hat{\xi} - \frac{2}{f_k(\hat{\xi})} \int_0^w f_k(w-w') \frac{\partial}{\partial \hat{\xi}} \nabla_\hat{\xi} \phi$$

Express $\frac{\partial}{\partial \hat{\xi}} \frac{\partial \phi}{\partial \xi}$ in terms of $\delta_{ij} = \frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j}$

\[ \phi \]

$dx_i$ and $dx_j$ are related at $w'$ by $A_{ij}(\hat{\xi}, w') = \frac{1}{f_k(w')} \frac{\partial x_i}{\partial \xi_j}$

\[ \phi \]

$$\Rightarrow \frac{\partial}{\partial \xi_j} \frac{\partial \phi}{\partial \xi_i} = \frac{\partial x_i}{\partial \xi_j} \frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j} = f_k(w') A_{ij}(\hat{\xi}, w') \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

(5)

$$A_{ij}(\hat{\xi}, w) = \frac{\partial \hat{\xi}_i}{\partial \xi_j} = \Delta \hat{\xi}_j - 2\int_0^w \frac{f_k(w-w') f_k(w')}{f_k(w)} \phi_{ij}(\xi(\hat{\xi}, w'), w') A_{kj}(\hat{\xi}, w')$$

(6)

This equation can be derived directly from the geodesic deviation equation of General Relativity, and is exact "in the limit of validity of the weak-field metric (1)" [Schneider p. 358].

Because of the remnant, $A_{ij}(\hat{\xi}, w)$ depending on $A_{ij}(\hat{\xi}, w')$ of all $w < w'$, it is not easy to use as is, so we need to make an approximation.
Now we make the "Born approximation," where we calculate the integral (5) instead along the unrestricted path, replacing $\iota$ with $\iota_0$.

$$r(\delta,w) = f_k(w) \delta \Rightarrow \frac{\partial}{\partial \theta_y} \frac{1}{f_k(w)} \frac{\partial}{\partial \theta_y} \Rightarrow A_{ij}(\delta,w) \approx S_{ij} \delta \quad (7)$$

$$\Rightarrow A_{ij}(\delta,w) \approx S_{ij} - 2 \int_0^w \frac{f_k(w-w')}{f_k(w)} \phi_{i0}(w,\delta, w') \phi_{0j}(w,\delta, w') \quad (8)$$

(Born apx)

Corrections to the Born apx are of order $\delta^2$. Since $\delta \ll 1$, we can use (8).

We can now define the deflection potential for cosmological shear

$$\phi(\delta,w) = 2 \int_0^w \frac{f_k(w-w')}{f_k(w)} \phi(w,\delta, w') \quad (9)$$

so that $A_{ij} = S_{ij} - \phi_{ij}$. Note Eq. (7b) for moving between

$$\phi_{i0} = \frac{\Theta \delta}{\theta_x \theta_y} \quad \text{and} \quad \phi_{0j} = \frac{\Theta^2 \delta}{\theta_x \theta_y}$$

and defining

$$A = \begin{bmatrix} 1 - \gamma_1 - \gamma_2 & \gamma_2 \\ -\gamma_2 & 1 - \gamma_1 - \gamma_1 \end{bmatrix}$$

we have

$$\gamma = \frac{1}{2} \gamma^2 = \frac{1}{2} (\gamma_{11} + \gamma_{22})$$

$$\gamma_1 = \frac{1}{2} (\gamma_{11} - \gamma_{22})$$

$$\gamma_2 = \gamma_{12}$$

$$\Theta = \gamma_1 + i \gamma_2 = \frac{1}{2} (\gamma_{11} - \gamma_{22}) + i \gamma_{12}$$

as before.