The metric perturbations $\phi$ are due to density perturbations $\delta$.

We assume that only matter density perturbations are significant.

In local coordinates $\nabla^2 \phi = \frac{8\pi G}{3m} \delta \Rightarrow$ In comoving coordinates

$$\nabla^2 \phi = \frac{8\pi G}{3} \hat{\delta} \Rightarrow \nabla^2 \phi = \frac{8\pi G}{3} \hat{\delta}$$

(9) $\Rightarrow \nabla^2 \phi = 2 \left( \sum_{w} \frac{f_k(w-w')f_k(w)}{f_k(w)} \right)^2 \phi f_k(w)$

We want to replace this with $\nabla^2 \phi = \nabla^2 \phi + \frac{\partial^2 \phi}{\partial x_3^2}$.

We did this in homework 10.3, where we integrated over a single lens, $\int \frac{2\phi}{2x_3^2} \mathrm{d}x_3 = \Delta \left( \frac{\partial \phi}{\partial x_3} \right)$.

The difference in $\frac{\partial \phi}{\partial x_3}$ between in front and behind the lens, assumed negligible sufficiently far from the lens. Now the situation is trickier, since we integrate over the entire light path, and do not assume that $\phi$ is negligibly anywhere. Also the two functions of $\phi$ play a role, since the integration is along lines, not at constant $d_\text{w}$. However, since $\phi$ is a perturbation, which takes both positive and negative values and should average to zero over the entire universe, we may expect some averaging out of the $\frac{\partial^2 \phi}{\partial x_3^2}$ part.

Bartelmann & Schneider, Phys. Rep. 310, 291 (2000) p.392: "We can now average it $\left[ \nabla^2 \phi \right]_{\Delta x_3}$, which involves only derivatives along the light path, because there average to zero in the limit to which we are working; the validity of this approximation has been verified with numerical simulations by White and Hu (1999)."'

Let's believe this and replace $\nabla^2 \phi$ with $\nabla^2 \phi = \frac{2}{3} H_0^2 \Omega_m \delta^1 \beta^5$

$\Rightarrow \nabla^2 \phi = 2 H_0^2 \Omega_m \int \frac{f_k(w')f_k(w-w')}{f_k(w)} \delta \left( f_k(w') \beta, w' \right) \equiv 2 H_0^2 \beta \phi (x, w)$

4) $x_3 = w\beta$, except that $\int \delta$ was assumed to be along the light ray, so that

$\phi (x, w) \text{ around } \phi (x_3 = w, x', w')$
The surface mass density \( H(\mathbf{\theta}, w) \) in direction \( \mathbf{\theta} \) depends on the distance \( w \) to the source. Consider a distribution \( P_w(w) \) of sources over different distances \( w \).

The effective surface mass density \( \Sigma(\mathbf{\theta}) \) is defined as the average over this distribution:

\[
\Sigma(\mathbf{\theta}) \equiv \frac{1}{\Omega_0} \int_{-\infty}^{\infty} P_w(w) \, \Sigma(\mathbf{\theta}, w) \, dw = \frac{3}{2} \frac{H_0^2}{\Omega_0} \int_{-\infty}^{\infty} P_w(w) \, \frac{P_k(w) f_k(w) f_k(w-w)}{f_k(w)} \, dw \frac{S(f_k(w), \mathbf{\theta})}{a(w)}
\]

\[
\approx g(w)
\]

Compared to (10), the factor \( \frac{f_k(w-w)}{f_k(w)} \) was replaced by the source-redshift weighted lens efficiency \( \frac{D_{\text{ls}}}{D_s} \) factor \( g(w) \).

Eq. (11) expresses \( \Sigma(\mathbf{\theta}) \) as a weighted projection of \( \Sigma \). This resembles the relation between the angular (galaxy number) density perturbation \( \frac{\delta n}{2\pi^2} \) and the BD density perturbation \( \tilde{\delta} \) from GSC part 1, 6.4.1:

\[
\frac{\delta n}{2\pi^2}(\mathbf{\theta}) = \frac{1}{V_5} \int_{-\infty}^{\infty} \Sigma(r) S(r) r^2 \, dr = \int_{0}^{\infty} dr \, q(r) \, S(r)
\]

where \( S(r) \) was the selection function, \( V_5 = \frac{\Omega_0^2}{\langle S \rangle} \), and \( q(r) = \frac{r}{V_5} S(r) r^2 \) is a weight function.

For the relation between the power spectrum of \( \delta n/\delta \) and \( \delta \) we obtained \( \delta n/\delta \) the small-angle flat sky limit (in 6.4.7.1) Limber's equation:

\[
\delta n(\mathbf{\theta}) \approx \frac{1}{L} \int_{0}^{\infty} dr \, S^2(r) \frac{P_\delta(L, \mathbf{\theta}, t(r))}{L}\delta \left( \frac{L}{r}, t(r) \right)
\]

\[
\frac{l^2}{2\pi^2} P_\delta(l) = \int_{0}^{\infty} dr \, r^2 \, q(r) \, P_\delta \left( \frac{L}{r}, t(r) \right)
\]

\[
\Rightarrow P_\delta(l) = \int_{0}^{\infty} dr \, r^2 \, q(r) \, P_\delta \left( \frac{L}{r}, t(r) \right)
\]
Luminosity equation (13) was derived for $k=0$ in GSC part 1. The generalization for $k \neq 0$ is:

$$P_{b}(l) = \int_{0}^{\infty} dw \frac{g(w)^2}{f_k(w)^2} P_\delta \left( \frac{l}{f_k(w)}, w \right)$$

(whence $w$ again indicates location on the light cone, i.e., implicitly also time)

Since (11) is like (12), except now $q(w) = \frac{3}{2} H_0^2 \Omega_m \frac{f_k(w) g(w)}{\alpha(w)}$

we can apply (14) to relate the power spectrum at $\Omega\Omega$ and $S(\rho)$:

$$P_{\Lambda}(l) = \frac{9}{4} H_0^2 \Omega_m^2 \int_{0}^{\infty} dw \frac{g^2(w)}{\alpha^2(w)} P_\delta \left( \frac{l}{f_k(w)}, w \right)$$

Thus the convergence power spectrum $P_{\Lambda}$ is a product of the density power spectrum $P_\delta$, of all matter, including dark matter, not just galaxies. A cosmological model and parameters, which predict $P_\delta$, predict thus also $P_{\Lambda}$.

The remaining issue is that we cannot observe $\kappa$ directly. Observations of galaxy ellipticities measure the reduced shear $\vartheta = \frac{\gamma}{1-h}$ for shear shear, mostly at $z > 1$, so that we can ignore the difference between $\vartheta$ and $\gamma$.

We shall now look at how to determine $P_{\Lambda}(l)$ from observations of shear $\gamma$. 
In §1.2 we had the integral relation between \( \phi \) and \( \tau \):

\[
\phi(\tau) = \frac{1}{4\pi} \int d^3\varphi' \, \chi(\varphi') \ln |\varphi' - \varphi| \tag{16}
\]

Denoting this twice for \( \chi_1 = \frac{1}{2} (\psi_{11} - \psi_{12}) \) and \( \chi_2 = \psi_{11} \) we had (excuse)

\[
\chi = \chi_1 + \chi_2 = \frac{1}{4\pi} \int d^3\varphi' \, \chi(\varphi') D(\varphi - \varphi') \tag{17}
\]

where

\[
D(\varphi) = \frac{\varphi_1^2 - \varphi_2^2 + 2i\varphi_1\varphi_2}{|\varphi|^4} = \frac{-1}{(\varphi_1 - i\varphi_2)^2} \tag{18}
\]

\[\begin{align*}
\text{Fourier transform} \\
\phi(\vec{\ell}) &= \int d^3\varphi \, e^{-i\vec{\varphi} \cdot \vec{\ell}} \phi(\varphi) \\
\chi_1(\vec{\ell}) &= \int d^3\varphi \, e^{-i\vec{\varphi} \cdot \vec{\ell}} \chi_1(\varphi) \\
\chi_2(\vec{\ell}) &= \int d^3\varphi \, e^{-i\vec{\varphi} \cdot \vec{\ell}} \chi_2(\varphi)
\end{align*}\]

\[
\chi(\vec{\ell}) = \frac{1}{4} \nabla^2 \phi(\vec{\varphi}) \quad \Rightarrow \quad \chi(\vec{\ell}) = -\frac{1}{2} \ell^2 \phi(\vec{\ell}) = -\frac{1}{2} (\ell_1^2 + \ell_2^2) \phi(\vec{\ell}) \tag{19a}
\]

\[
\chi_1(\vec{\ell}) = \frac{1}{2} (\psi_{11} - \psi_{12}) \quad \Rightarrow \quad \chi_1(\vec{\ell}) = -\frac{1}{2} (\ell_1^2 - \ell_2^2) \phi(\vec{\ell}) \tag{19b}
\]

\[
\chi_2(\vec{\ell}) = \psi_{11} \quad \Rightarrow \quad \chi_2(\vec{\ell}) = -\ell_1 \ell_2 \phi(\vec{\ell}) \tag{19c}
\]

Eq. (17) is a convolution in \( \varphi \) space \( \Rightarrow \) it becomes a product in Fourier space

\[
\chi(\vec{\ell}) = \chi_1(\vec{\ell}) + \chi_2(\vec{\ell}) = \frac{D(\vec{\ell}) \chi(\vec{\ell})}{\pi} \quad \text{where} \quad D(\vec{\ell}) = \pi \frac{\ell_1^2 - \ell_2^2 + 2i\ell_1\ell_2}{\ell^2} \tag{20}
\]

(Excuse; easiest to show using (19)).

We have \( D(\vec{\ell}) D(\vec{\ell})^* = \pi^2 \) (excuse) \( \Rightarrow \)

\[
\chi(\vec{\ell}) = \frac{D^*(\vec{\ell}) \chi(\vec{\ell})}{\pi} \tag{21}
\]

These relations hold only for \( \vec{\ell} \neq 0 \), \( D(0) \) is undefined.