Statistical homogeneity \( \Rightarrow \) different Fourier modes are uncorrelated

\[
\langle \psi(l)^* \psi(l') \rangle = (2\pi)^2 \delta^2_0 (l-l') P_q(l)
\]

Using (19),

\[
\begin{align*}
\langle \psi(l)^* \psi(l') \rangle &= \frac{1}{l^4} \langle \psi(l)^* \psi(l') \rangle \Rightarrow P_q(l) = \frac{1}{l^4} P_q(l) \\
\langle \psi_1(l)^* \psi_1(l') \rangle &= \frac{1}{l^2} \langle \psi(l)^* \psi(l') \rangle \Rightarrow P_{g1}(l) = \frac{1}{l^2} \langle \psi(l)^* \psi(l') \rangle \\
\langle \psi_2(l)^* \psi_2(l') \rangle &= \frac{1}{l^2} \langle \psi(l)^* \psi(l') \rangle \Rightarrow P_{g2}(l) = \frac{1}{l^2} \langle \psi(l)^* \psi(l') \rangle \\
\end{align*}
\]

\[
P_{g1}(l) + P_{g2}(l) = \frac{1}{l^2} \left[ \langle \psi(l)^* \psi(l') \rangle + \langle \psi(l)^* \psi(l') \rangle \right] P_q(l) = \frac{1}{l^4} P_q(l) = P_q(l)
\]

\( \psi \) and \( \chi \) are scalar quantities. Statistical isotropy \( \Rightarrow P_q(l) = P_q(l) \) and \( P_{g1}(l) = P_{g1}(l) \)

\( y \) is a polar quantity. It's components \( y_1 \) and \( y_2 \) depend on the orientation of the coordinate system. The dependence on the coordinate system breaks the isotropy at \( y_1 \) and \( y_2 \), and hence \( P_{g1}(l) \) and \( P_{g2}(l) \) depend on the direction of the 2D wave vector \( \vec{l} \).

So far we have worked only with Fourier transforms and power spectra of real quantities, to be on more familiar ground. But we can also define a power spectrum for a complex \( y = y_1 + iy_2 \). Do this for correlator

\[
\langle \psi_1(l)^* \psi_2(l') \rangle = \langle \psi_1(l)^* \psi_2(l') \rangle \Rightarrow y_1(l) \text{ and } y_2(l) \text{ are correlated; this correlator is real (since } \langle \psi(l)^* \psi(l') \rangle \text{ is)}
\]

\[
\begin{align*}
\langle \psi(l)^* \psi(l') \rangle &= \langle \psi_1(l)^* \psi_2(l') + iy_2(l)^* \psi_2(l') \rangle \\
&= \langle \psi_1(l)^* \psi_1(l') \rangle + i \langle \psi_1(l)^* \psi_2(l') \rangle - i \langle \psi_2(l)^* \psi_1(l') \rangle + \langle \psi_2(l)^* \psi_2(l') \rangle \\
\end{align*}
\]

These are equal (complex conjugates of a real quantity), so they cancel.
Defining $P_y(l^2)$ by $\langle \gamma(l^2) \gamma(l'^2) \rangle = (2\pi)^2 \delta^2(l-l') P_y(l^2)$ we have shown

$$P_y(l^2) = P_{y_1}(l^2) + P_{y_2}(l^2) = P_x(l^2)$$  \hspace{1cm} (28) \hspace{1cm} \because P_y(l^2) \text{ is isochoric}$$

In terms of length and direction of the 2D wave vector $\vec{\ell}$

$$L_1 = \ell \cos \varphi, \quad L_2 = \ell \sin \varphi.$$  

$$P_{y_1}(l^2) = \left( \frac{\ell_1^2 - \ell_2^2}{l^2} \right)^2 P_x(l^2) = \left( \cos^2 \varphi_0 - \sin^2 \varphi_0 \right) P_x(l^2) = \cos^2 2\varphi_0 P_x(l^2)$$

$$P_{y_2}(l^2) = \left( \frac{\ell_1^2 + \ell_2^2}{l^2} \right)^2 P_x(l^2) = \left( \cos^2 \varphi_0 + \sin^2 \varphi_0 \right) P_x(l^2) = \sin^2 2\varphi_0 P_x(l^2)$$  \hspace{1cm} (28a)

Or directly for $\gamma(l^2) = \gamma_1(l^2) + i\gamma_2(l^2)$:

$$\gamma(l^2) = \frac{\ell_1^2 - \ell_2^2 + 2i\ell_1 \ell_2}{l^2} \gamma_1(l^2) = \frac{(l_1 + i l_2)^2}{l^2} \gamma_1(l^2) = e^{2i\varphi_0} \gamma_1(l^2)$$  \hspace{1cm} (29)

$$\Rightarrow \quad P_y(l^2) = |e^{2i\varphi_0}|^2 P_x(l^2) = P_x(l^2)$$

Here $L = \sqrt{l_1^2 + l_2^2} = |l^2| = |l_1 + i l_2|$ is both the length of the vector $\vec{l}$

and its modulus (absolute value) of the complex number $l_1 + i l_2$.

Thus we can relate the isochoric power spectrum $P_x(l^2)$ to the shear power spectrum $P_y(l^2)$ — they are the same!

However, what we can get directly from observation, are the shear correlation functions rather than their power spectra.
5.4 Shear Correlation Functions

- We define the tangential and cross component of shear for a pair of points (galaxy images) using their separation line as the reference direction:

\[
\gamma_t = -\text{Re}(ye^{-i2\phi}) \\
\gamma_x = -\text{Im}(ye^{-i2\phi})
\]

Note that since \(e^{i2\phi} = e^{-i2(\phi + \pi)}\), it doesn't matter which way the separation direction is defined.

- We define the correlation functions:

\[
\langle \gamma_t(y_1) \gamma_t(y_2) \rangle = \langle \gamma_t \gamma_t \rangle (b) = \langle \gamma_t \gamma_t \rangle (0) \\
\langle \gamma_x(y_1) \gamma_x(y_2) \rangle = \langle \gamma_x \gamma_x \rangle (b) = \langle \gamma_x \gamma_x \rangle (0) \\
\langle \gamma_t(y_1) \gamma_x(y_2) \rangle = \langle \gamma_t \gamma_x \rangle (b) = \langle \gamma_t \gamma_x \rangle (0) = \tilde{S}_x(0)
\]

\[
\tilde{S}_x(0) = \langle \gamma_t \gamma_t \rangle (0) + \langle \gamma_x \gamma_x \rangle (0)
\]

- These are evaluated from the data simply by using galaxy image ellipticities as shear estimates: For each bin \(\Delta\theta\) in \((\theta + \Delta\theta/2)\), find all \(N\) image pairs whose separation falls in this range, and calculate the average:

\[
\tilde{S}_x(n) = \frac{1}{N} \sum_{i,j} (E_{t,i}E_{t,j} + E_{x,i}E_{x,j})
\]

Note that the tangential and cross components \(E_t, E_x\) for each galaxy image are defined differently for each pair and it is part of...
Meaning at $y_t$ and $y_x$ in pictures:

$y_t > 0$
$y_x = 0$

$y_t < 0$
$y_x = 0$

$y_t = 0$
$y_x = 0$

The reason for the sign in the definition of $y_t$: This way it is typically expected to be positive; the more moving of the pair of galaxies based on the more distant one in the tangential direction. (Without the - sign we should call $y_t$ the radial component.)

I didn't bother to check the sign at $y_x$ in the third picture; it will depend on how we define the coordinate axes: if we define a right-handed 3D coordinate system, it depends on whether we choose the third axis as pointing towards or away from the observer.

Parity symmetry $\Rightarrow \delta_x(\vec{0})$ is expected to vanish, since in a parity transformation (mirror universe) $y_t \to y_t$, but $y_x \to -y_x$.

We should now relate $\delta_t(\vec{0})$ to $P_x(\vec{0})$.

I don't know if that's a clever way to do this directly with no complex skewers.

I did this by falling back to $y_1$ and $y_2$, defining four correlation functions

$\delta_1(\vec{0}) = \langle y_1(\vec{0}_1) y_1(\vec{0}_2) \rangle$

where $\vec{0} = \vec{0}_2 - \vec{0}_1$

$\delta_2(\vec{0}) = \langle y_2(\vec{0}_1) y_2(\vec{0}_2) \rangle$

These correlation functions are not expected to be isotropic, since the components depend on the orientation of the old system.

$\delta_{12}(\vec{0}) = \langle y_1(\vec{0}_1) y_2(\vec{0}_2) \rangle$

But statistical homogeneity $\Rightarrow$ they depend only on the separation $\vec{0}$.

(On next page I call $\vec{0}_1 = \vec{0}_0$ and $\vec{0}_2 = \vec{0}_0 + \vec{0}$.)
The correlation functions \( \mathbf{S}_1(\mathbf{q}) \) and \( \mathbf{S}_2(\mathbf{q}) \) are Fourier transforms of their power spectra \( P_{S_1}(l) \) and \( P_{S_2}(l) \).

We have to deal with two directions of 2D vectors:
- that at \( \mathbf{q} \)
- and that at \( \mathbf{q}' \)

\[ \mathbf{q}' = \mathbf{q} - \mathbf{q} \text{ is the angle between } \mathbf{q} \text{ and } \mathbf{q}' \]

\[ \Rightarrow \mathbf{q} \cdot \mathbf{q}' = 0 \text{ as } \mathbf{q}' \]

\[ \mathbf{S}_1(\mathbf{q}) = \langle \psi_1(\mathbf{q}+\mathbf{0}) \psi_1(\mathbf{q}+\mathbf{q}') \rangle = \frac{1}{(2\pi)^2} \int d^2 l \ P_{S_1}(l) \ e^{i \mathbf{l} \cdot \mathbf{q}'} = = \frac{1}{(2\pi)^2} \int d^2 l \ P_{S_1}(l) \ e^{i \mathbf{l} \cdot \mathbf{q}'} \]

Using \( \int d^2 l = 2 \pi \int_0^{2\pi} \int_0^\infty \rho \ d\rho \ d\phi = 2 \pi \int_0^{2\pi} \int_0^\infty \rho^2 \sin \phi \ d\rho \ d\phi \) and the integral representation of Bessel functions

\[ \text{Bessel function } J_0(x) = \frac{(-1)^n}{n!} \int_0^\infty e^{i x \cos \theta \sin \phi} \sin^2 \phi \ d\phi = \frac{(-1)^n}{2\pi} \int_0^{2\pi} e^{i x \cos \theta \sin \phi} \sin^2 \phi \ d\phi \]  

one finds (accurate)

\[ \mathbf{S}_1(\mathbf{q}) = \frac{1}{(2\pi)^2} \int d^2 l \ P_{S_1}(l) \ [J_0(l) + (\cos^2 \phi - \sin^2 \phi) J_1(l)] \]

\[ \mathbf{S}_2(\mathbf{q}) = \frac{1}{(2\pi)^2} \int d^2 l \ P_{S_2}(l) \ [J_0(l) + (\sin^2 \phi - \cos^2 \phi) J_1(l)] \]

Also

\[ \mathbf{S}_{12}(\mathbf{q}) = \langle \psi_1(\mathbf{q}+\mathbf{0}) \psi_2(\mathbf{q}+\mathbf{q}') \rangle = \frac{1}{(2\pi)^4} \int d^2 l_1 \int d^2 l_2 \ e^{-i \mathbf{l}_1 \cdot \mathbf{q}'} e^{-i \mathbf{l}_2 \cdot \mathbf{q}'} \langle \psi_1(\mathbf{l}_2+\mathbf{q}) \psi_2(\mathbf{l}_1+\mathbf{q}') \rangle \]

\[ = \ldots = \frac{1}{(2\pi)^4} \int d^2 l_1 \ P_{S_1}(l_1) \cdot 2 \sin \phi \cos \phi \cdot J_4(l_1) \text{ spherical harmonics} \]

and \( \mathbf{S}_{21}(\mathbf{q}) = \langle \psi_2(\mathbf{q}+\mathbf{0}) \psi_1(\mathbf{q}+\mathbf{q}') \rangle = \langle \psi_2(\mathbf{q}+\mathbf{0}) \psi_1(\mathbf{q}+\mathbf{q}') \rangle = \langle \psi_1(\mathbf{q}+\mathbf{0}) \psi_2(\mathbf{q}+\mathbf{q}') \rangle = \langle \psi_1(\mathbf{q}+\mathbf{0}) \psi_2(\mathbf{q}+\mathbf{q}') \rangle \)

\[ \mathbf{S}_{12}(-\mathbf{q}) = \mathbf{S}_{12}(\mathbf{q}) \text{ since } \sin^2 \phi \cos^2 \phi \text{ is invariant in } q + \phi + \pi \]

\( (\mathbf{q} + \phi + \pi) \)
\[ y e^{-12q} = (y_1 + iy_2)(\cos 2q - i\sin 2q) = y_1 \cos 2q + y_2 \sin 2q + i [-y_1 \sin 2q + y_2 \cos 2q] \]

\[ \Rightarrow \]

\[ y_1 = -\text{Re} (y e^{-12q}) = -y_1 \cos 2q - y_2 \sin 2q \]

\[ y_2 = -\text{Im} (y e^{-12q}) = y_1 \sin 2q - y_2 \cos 2q \]

\( (33) \)

\[ \Rightarrow \quad \langle y_1 y_2 \rangle (\beta) = \cdots = \frac{1}{4\pi} \int_{0}^{\infty} \frac{d\ell}{\ell} P_0 (\ell) \cdot [J_0 (\ell \beta) + J_1 (\ell \beta)] \]

\[ \langle y_1 y_2 \rangle (\beta) = \cdots = \frac{1}{4\pi} \int_{0}^{\infty} \frac{d\ell}{\ell} P_0 (\ell) \cdot [J_0 (\ell \beta) - J_1 (\ell \beta)] \]

\[ \]

\[ \xi_+ (\beta) = \langle y_1 y_2 \rangle + \langle y_2 y_1 \rangle = \frac{1}{2\pi} \int_{0}^{\infty} \frac{d\ell}{\ell} J_0 (\ell \beta) P_0 (\ell) \]

\[ \xi_- (\beta) = \langle y_1 y_2 \rangle - \langle y_2 y_1 \rangle = \frac{1}{2\pi} \int_{0}^{\infty} \frac{d\ell}{\ell} J_1 (\ell \beta) P_0 (\ell) \]

\( (34) \)

Also \( \xi_0 (\beta) = \langle y_1 y_2 \rangle (\beta) = \cdots = 0 \)

We can insert these equations using the Bessel function closure (orthogonality) equation

\[ \int_{0}^{\infty} J_\alpha (x) J_\beta (x \alpha') \, dx = \frac{1}{2\pi} \delta_D (x - \alpha') \]  

\( (35) \)

\[ 2\pi \int_{0}^{\infty} \xi_+ (\ell) J_0 (\ell \beta) \, d\ell = \int_{0}^{\infty} \frac{d\ell'}{\ell'} \int_{0}^{\infty} \frac{d\ell}{\ell} \, J_0 (\ell \beta) J_0 (\ell \beta') P_0 (\ell') \]

\[ = \int_{0}^{\infty} \frac{d\ell}{\ell} \int_{0}^{\infty} \frac{d\ell'}{\ell'} \, J_0 (\ell \beta) J_0 (\ell \beta') \, d\ell = P_0 (\ell) \]  

\( (36a) \)

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\[ 2\pi \int_{0}^{\infty} \xi_- (\ell) J_0 (\ell \beta) \, d\ell = \cdots = P_0 (\ell) \]  

\( (36b) \)
Thus the two shear correlation functions \( \Sigma_+(\theta) \) and \( \Sigma_-(\theta) \) are not independent (as the two components of shear are both derived from the same scalar quantity, the deflection potential \( \Psi \)). We can express them in terms of each other by resolving (36b) into (34a) and (36a) into (34b). Scherbius (p.364) gives the results:

\[
\begin{align*}
\Sigma_+(\theta) &= \frac{1}{2\pi} \int_0^\infty \int_0^\infty J_0(\mu) \mu P_+(\mu) d\mu d\theta' \\
&= \frac{1}{2\pi} \int_0^\infty \int_0^\infty J_0(\mu) J_0(\mu') d\mu d\theta' \Sigma_+(\theta') \\
&= \frac{1}{2\pi} \int_0^\infty \int_0^\infty J_0(\mu) J_0(\mu') \Sigma_+(\theta') \left( 1 - \frac{\theta'^2}{\theta^2} \right) \\
&= \frac{1}{2\pi} \int_0^\infty \int_0^\infty J_0(\mu) J_0(\mu') \Sigma_+(\theta') \left( 1 - \frac{\theta'^2}{\theta^2} \right) \\
\Sigma_-(\theta) &= \frac{1}{2\pi} \int_0^\infty \int_0^\infty J_0(\mu) \mu P_-(\mu) d\mu d\theta' \\
&= \frac{1}{2\pi} \int_0^\infty \int_0^\infty J_0(\mu) J_0(\mu') d\mu d\theta' \Sigma_-(\theta') \\
&= \frac{1}{2\pi} \int_0^\infty \int_0^\infty J_0(\mu) J_0(\mu') \Sigma_-(\theta') \left( 1 - \frac{\theta'^2}{\theta^2} \right) \\
&= \frac{1}{2\pi} \int_0^\infty \int_0^\infty J_0(\mu) J_0(\mu') \Sigma_-(\theta') \left( 1 - \frac{\theta'^2}{\theta^2} \right)
\end{align*}
\]

I haven't so far managed to do this. Tools for doing the integral over the Bessel function product include:

- Bessel function recursion formulas, e.g., \( J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \)

- The closure relation (33), which can be used to give the first term in (37a,b)

- Gradsteyn & Ryzhik integral 6.512.3:

\[
\int_0^\infty J_\alpha(x) J_{\alpha-1}(x) dx = \begin{cases} 
0 & \text{if } \alpha < \beta \\
\frac{1}{2\beta} & \text{if } \alpha = \beta \\
\frac{\sin \alpha}{\alpha} & \text{if } \alpha > \beta
\end{cases}
\]

which can be used to cut the integral \( \int_0^\infty \int_0^\infty J_0(\mu) J_0(\mu') d\mu d\theta' \) to \( \int_0^\infty \int_0^\infty J_0(\mu) J_0(\mu') d\mu d\theta' \) or \( \int_0^\infty \int_0^\infty J_0(\mu) J_0(\mu') d\mu d\theta' \)

(The exact forms of (37a,b) are maybe not that important; just the principle is.)