As we have seen, the shear field $\gamma^i(\vec{r})$ cannot be an arbitrary complex (or 2-component real) field, but it satisfies certain constraint relations, which are due to it arising from a single scalar potential $\phi(\vec{r})$. The constraint can be expressed in terms of a vector field $\vec{A}$.

In this section we deal with lots of derivatives of $\phi(\vec{r})$; so we introduce a further simplified notation for them (dropping the comma $,$):

$$\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \phi_{11} = \phi_{12} \, \text{ etc.}$$

Remember:

$$\mathbf{H} = \frac{1}{2} \nabla^2 \phi = \frac{1}{2} (\phi_{11} + \phi_{22}) \quad \phi_{11} = \frac{1}{2} (\phi_{11} - \phi_{12}) \quad \phi_{22} = \frac{1}{2} (\phi_{11} - \phi_{12})$$

We define $\vec{A}$ as $\nabla \mathbf{H}$ (which guarantees it is a vector field):

$$\nabla \mathbf{H} = \begin{pmatrix} H_{11} \\ H_{12} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \phi_{11} + \phi_{12} \\ \phi_{12} + \phi_{22} \end{pmatrix} = \begin{pmatrix} \gamma_{11} + \gamma_{22} \\ \gamma_{21} - \gamma_{12} \end{pmatrix} = \vec{A}(\vec{r}) \quad (38)$$

This gives a local relation between (derivatives of) $\mathbf{H}$ and $\gamma$.

$$\nabla^2 \mathbf{H} = \nabla \cdot \vec{A} \quad \text{and} \quad \nabla \times \vec{A} = \nabla \times \nabla \mathbf{H} = 0$$

Thus the shear field satisfies the constraint $\nabla \times \vec{A} = 0$ (39).

Another constraint relation, of the same origin, for the shear field in (from Eq. 36)

$$\int \delta \phi \, F_+ (\theta) \, J_x (\theta) = \int \delta \phi \, F_- (\theta) \, J_y (\theta)$$ (40)
However, the measured shear field based on galaxy image ellipticities does not necessarily satisfy the constraints (39) or (40); and these also other relations from §5.4 do not have to hold for it. This may be due to a number of reasons:

1) statistical error ("noise") due to ellipticities of sources

2) systematic error (intrinsic correlations of source ellipticities or between source ellipticity and shear)

3) higher order effects (Bern apx not valid; Schneider mentions also "clustering of sources")

Also, e.g., instrumental effects may contribute to statistical/systematic errors.

The difference between statistical and systematic error is that the former can be reduced by increasing the size of the survey (more galaxy images).

We can provide a description of the measured, unconstrained, shear field in terms of a potential by introducing a complex potential

\[ \phi = \phi^E + i\phi^B \]

where \( \phi^E, \phi^B \) are real

Now \( \phi^B \) provides the missing degree of freedom. Defining

\[ y = \frac{1}{2}(\psi_{11} - \psi_{22}) + i\psi_{12} = \frac{1}{2}(\psi_{11}^E - \psi_{22}^E) - \psi_{12}^B + i\left[\psi_{12}^E + \frac{1}{2}(\psi_{11}^B - \psi_{22}^B)\right] \]

\[ y_1 = \Re y = \frac{1}{2}(\psi_{11}^E - \psi_{22}^E) - \psi_{12}^B \quad \quad y_2 = \Im y = \psi_{12}^E + \frac{1}{2}(\psi_{11}^B - \psi_{22}^B) \]

We also define a complex conjugate \( \bar{\psi} = \psi^E + i\psi^B \) by

\[ \bar{\psi} = \frac{1}{2}V^2\psi \Rightarrow \bar{\psi}^E = \frac{1}{2}V^2\psi^E \quad \bar{\psi}^B = \frac{1}{2}V^2\psi^B \]
We deduce the vector field \( \mathbf{u}_y(s) \) in terms of the real \( y_1 \) and \( y_2 \); so the definition is still

\[
\mathbf{u}_y(s) = \left( \frac{y_{11} + y_{22}}{\gamma_{11} - \gamma_{12}}, \right)
\]

so that \( \mathbf{u}_y \) is real. The way \( y_1 \) and \( y_2 \) were defined in (42) means that now \( \mathbf{u}_y \neq \nabla \lambda \) and we get \( \nabla \times \mathbf{u}_y \neq 0 \). Since \( \mathbf{u}_y \) is a 2D vector field, \( \nabla \times \mathbf{u}_y \) is a scalar:

\[
\nabla \times \mathbf{u}_y = u_{21} - u_{12}
\]

Calculating (exercise)

\[
\begin{align*}
\nabla \cdot \mathbf{u}_y &= u_{21} + u_{22} = y_{11} + y_{22} + y_{21} - y_{12} = \cdots = \nabla^2 \mu^E \\
\nabla \times \mathbf{u}_y &= u_{21} - u_{12} = \cdots = \nabla^2 \mu^B
\end{align*}
\]

1. We divide the measured shear field \( \gamma \) into two parts (modes):

E) The part with \( \nabla \times \mathbf{u}_y = 0 \), which has the properties expected of shear, and thus provides our estimate of the true shear.

B) The part with \( \nabla \cdot \mathbf{u}_y = 0 \), which presumably is due to error (systematic + statistical) in shear measurement and higher-order effects.

The error & higher-order effects seen in B presumably contribute in similar magnitude to be E mode; thus the measured B mode provides an error estimate for our estimate of the true shear.

For small surveys, statistical error is expected to dominate the B mode.

For larger surveys, a stronger B mode has been observed than expected from statistical error or higher-order effects; the natural conclusion is that it must be due to systematic effects, like correlations between some allipathies, or between some allipathies and shear.

\( * \) But \( \mathbf{u}_y(s) \), as defined by (41), is still a vector field (exercise).
The division of measured shear into \(E\) and \(B\) mode is simple in Fourier space:

\[
\begin{align*}
\gamma^E (\ell) & = -\frac{1}{2} (l_z^2 - l_s^2) \psi^E (\ell) = -\frac{1}{2} l_z^2 \cos 2\ell_z \cdot \psi^E (\ell) = \cos 2\ell_z \cdot \eta^E (\ell) \\
\gamma^B (\ell) & = l_z l_s \psi^B (\ell) = \frac{1}{2} l_z^2 \sin 2\ell_z \cdot \psi^B (\ell) = -\sin 2\ell_z \cdot \eta^B (\ell) \\
\gamma^E (\ell) & = -l_z l_s \psi^E (\ell) = -\frac{1}{2} l_z^2 \sin 2\ell_z \cdot \psi^E (\ell) = \sin 2\ell_z \cdot \eta^E (\ell) \\
\gamma^B (\ell) & = -\frac{1}{2} (l_z^2 - l_s^2) \psi^B (\ell) = -\frac{1}{2} l_z^2 \sin 2\ell_z \cdot \psi^B (\ell) = \cos 2\ell_z \cdot \eta^B (\ell)
\end{align*}
\]

\[
\Rightarrow \ \gamma^E (\ell) = e^{i2\ell_z} \eta^E (\ell) \quad \text{and} \quad \gamma^B (\ell) = e^{i2\ell_z} \eta^B (\ell)
\]

\[
\gamma (\ell) = \gamma^E (\ell) + \gamma^B (\ell) = e^{i2\ell_z} \left[ \eta^E (\ell) + i \eta^B (\ell) \right]
\]

\[
\Rightarrow \begin{align*}
\eta^E (\ell) & = \text{Re} \left[ e^{-i2\ell_z} \gamma (\ell) \right] = \cos 2\ell_z \cdot \gamma_1 (\ell) - \sin 2\ell_z \cdot \gamma_2 (\ell) \\
\eta^B (\ell) & = \text{Im} \left[ e^{-i2\ell_z} \gamma (\ell) \right] = -\sin 2\ell_z \cdot \gamma_1 (\ell) + \cos 2\ell_z \cdot \gamma_2 (\ell)
\end{align*}
\]

\[\therefore \text{From measured } \gamma, \text{ obtain } \eta^E \text{ and } \eta^B \text{ using (48), and then } \gamma^E \text{ and } \gamma^B \text{ using (46) or (47).}\]

How do the \(E\) and \(B\) modes look on the sky:

**Figure:** \(E\)-mode and \(B\)-mode patterns of shear. The top left pattern is caused by a mass overdensity and the top right pattern by mass underdensity.

The \(B\)-mode patterns cannot be caused by gravitational lensing.
We define the E and B mode power spectra as

\[
\begin{align*}
\langle A^E (l) \times A^E (l') \rangle &= \frac{1}{2} l^4 \langle A^E^* \times A^E \rangle = (2\pi)^4 \delta^2 (l - l') P_E (l) \\
\langle A^B (l) \times A^B (l') \rangle &= \frac{1}{2} l^4 \langle A^B^* \times A^B \rangle = (2\pi)^4 \delta^2 (l - l') P_B (l) \\
\langle A^E (l) \times A^B (l') \rangle &= \frac{1}{2} l^4 \langle A^E^* \times A^B \rangle = (2\pi)^4 \delta^2 (l - l') P_{EB} (l) = 0
\end{align*}
\]  

(49)

The cross spectrum \( P_{EB} (l) \) vanishes for parity-symmetric shear fields — we assume this. Thus \( \langle A^E^* \times A^B \rangle = 0 \Rightarrow \) all \( \langle A^E^* A^B \rangle = 0 \)  

(50)

For shear correlations we get

\[
\begin{align*}
\langle \gamma^E (l) \times \gamma^E (l') \rangle &= \langle [(\gamma^E (l) + \gamma^B (l)) \times [\gamma^E (l') + \gamma^B (l')] \rangle = \langle \gamma^E^* \times \gamma^E \rangle + \langle \gamma^B^* \times \gamma^B \rangle
\end{align*}
\]

\[
\begin{align*}
P_{\gamma^E} (l) &= P_{\gamma^E} (l) + P_{\gamma^E} (l) \\
P_{\gamma^B} (l) &= P_{\gamma^B} (l) + P_{\gamma^B} (l)
\end{align*}
\]

(49) \( \Rightarrow \)

\[
\begin{align*}
P_{\gamma^E} (l) &= \omega^2 2q_E P_E (l) & P_{\gamma^E} (l) &= \delta \omega^2 2q_E P_E (l) \\
P_{\gamma^B} (l) &= \delta^2 2q_B P_B (l) & P_{\gamma^B} (l) &= \omega^2 2q_B P_B (l)
\end{align*}
\]  

(51)

Shear Correlation Functions with E and B modes

We have now to repeat the calculations from §5.4, including now also the B modes. The actual calculations are largely left as an exercise. If one has already done the calculations in §5.4, then only the E-mode part, and only the B-mode part requires new calculation. Since we assumed parity symmetry, there will be no EB cross term. The outline and results:

\[ \mathcal{S}_1(\delta) = \langle \chi_i(\theta_o) \chi_i(\theta_x, \phi) \rangle = \frac{1}{(2\pi)^2} \int d^2\ell \frac{P_Y(\ell) \cos \theta}{ \ell \cdot \hat{\delta} } = \ldots \]

\[ P_{E} + P_{B} = \cos^2 2\phi \cdot P_{E}(\ell) + \sin^2 2\phi \cdot P_{B}(\ell) \]

\[ \mathcal{S}_2(\delta) = \ldots = \frac{1}{(2\pi)^2} \int d^2\ell \left[ P_{E}(\ell) \left[ J_0(10) + (1 - \cos 2\phi - \sin 2\phi) J_4(10) \right] + P_{B}(\ell) \left[ J_0(10) + (1 + \cos 2\phi + \sin 2\phi) J_4(10) \right] \right] \]

\[ \mathcal{S}_{12}(\delta) = \ldots = \frac{1}{(2\pi)^2} \int d^2\ell \left[ P_{E}(\ell) - P_{B}(\ell) \right] \cdot 2 \sin 2\phi \cos 2\phi \cdot J_4(10) \]

With \( \chi_e = -\chi_1 \cos 2\phi - \chi_2 \sin 2\phi \), \( \chi_x = \chi_1 \sin 2\phi - \chi_2 \cos 2\phi \)

we then get

\[ \langle \chi_i \chi_i \rangle(\delta) = \cos^2 2\phi \cdot \mathcal{S}_1(\delta) + 2 \sin 2\phi \cos 2\phi \cdot \mathcal{S}_{12}(\delta) + \sin^2 2\phi \cdot \mathcal{S}_2(\delta) = \ldots \]

\[ = \frac{1}{(2\pi)^2} \int d^2\ell \left[ \left[ P_{E}(\ell) + P_{B}(\ell) \right] J_0(10) + \left[ P_{E}(\ell) - P_{B}(\ell) \right] J_4(10) \right] \]

\[ \langle \chi_i \chi_x \rangle(\delta) = \ldots = \frac{1}{(2\pi)^2} \int d^2\ell \left[ \left[ P_{E}(\ell) + P_{B}(\ell) \right] J_0(10) - \left[ P_{E}(\ell) - P_{B}(\ell) \right] J_4(10) \right] \]

\[ \mathcal{S}_4(\delta) = \langle \chi_i \chi_i \rangle + \langle \chi_i \chi_x \rangle = \frac{1}{2\pi} \int d^2\ell \left[ P_{E}(\ell) + P_{B}(\ell) \right] J_0(10) \]

\[ \mathcal{S}_4(\delta) = \langle \chi_i \chi_i \rangle - \langle \chi_i \chi_x \rangle = \frac{1}{2\pi} \int d^2\ell \left[ P_{E}(\ell) - P_{B}(\ell) \right] J_4(10) \]

Since \( \mathcal{S}_4 \) depends on \( P_{E} + P_{B} \) and \( \mathcal{S}_4 \) on \( P_{E} - P_{B} \), they can no longer be obtained from each other, but are independent.
Using the Bessel function closure relation (35)

\[ 2\pi \int_0^\infty \xi_+ \xi_0 + \xi_- \xi_0 \, d\theta = P_E(\ell) + P_B(\ell) \]

\[ 2\pi \int_0^\infty \xi_- \xi_0 - \xi_+ \xi_0 \, d\theta = P_E(\ell) - P_B(\ell) \]

\[ P_E(\ell) = \pi \int_0^\infty \left[ \xi_+ \xi_0 \, d\theta + \xi_- \xi_0 \, d\theta \right] \]

\[ P_B(\ell) = \pi \int_0^\infty \left[ \xi_+ \xi_0 \, d\theta - \xi_- \xi_0 \, d\theta \right] \]

The two power spectra \( P_E(\ell) \) and \( P_B(\ell) \) can be obtained from the measured correlation functions \( \xi_+ (\ell) \) and \( \xi_- (\ell) \).

However, this involves integrating \( \xi_\ell \ell \) over infinite range. For practical work, these are methods (e.g., "aperture averages") designed for finite sumups; but we don't have time to cover them in this course (this year, maybe they can be included when the course is continued next time).

THE END