§2.2 Point-Mass Lens

Violate earlier assumptions \( |l| < a_{\text{max}} \) (or \( \Xi \) smooth)

From where we started:

\[
\Delta = \frac{D_{\ell}}{D_d} \frac{yGM}{c^2} = \frac{D_{\ell}}{D_d D_d} \frac{yGM}{c^2} = \frac{m}{v^2} = \frac{\phi}{F}
\]

\( \Rightarrow \beta = \eta - \alpha = \left[ 1 - \frac{(\phi)^2}{F} \right] \eta \)

Choose \( \beta \) positive; \( \eta \) may have either sign

\( \begin{align*}
X = \frac{\phi}{\phi_E}, & \quad y = \frac{\beta}{\phi_E} \\
\Rightarrow & \quad y = \left( 1 - \frac{1}{X^2} \right) x = x - \frac{1}{X} \Rightarrow x = \frac{1}{2} (y \pm \sqrt{y^2 + 1}) \\
X_+ > y, & \quad |X| > 1, \quad X_+ \geq 1 \\
-1 < X < 0 & \\
\end{align*} \)

\( \forall \phi, \quad \text{two image each side of lens and same} \)

For \( \beta = \phi_E, \quad \phi = \frac{1}{2} (1 \pm \sqrt{5}) \phi_E = \begin{cases} 1.62 \phi_E \\ -0.62 \phi_E \end{cases} \)

\( \mu(\phi) = \frac{m}{\phi^2} = \frac{(\phi_E)^2}{\phi}\quad \text{outside the mass distribution} \)

Magnification

\[
\mu = \frac{1}{\Delta A} = \frac{1}{(1-\eta)(1+\eta-2\eta)} = \frac{1}{1-\eta^2} = \frac{1}{1-x^2}
\]

\( \mu = 0 \quad \text{outside the mass distribution} \)

\[
= \pm \frac{1}{x^2} \left( \frac{y}{\sqrt{y^2+1}} + \sqrt{y^2+1} \right) \pm 2 \\
\quad \text{exercise, may be tedious} \\
\quad > 2 \quad (\rightarrow 2 \text{ as } y \rightarrow \infty); \quad \text{so that } \mu \rightarrow 1 \text{ and } \mu \rightarrow 0 \)

\( X_+ > 1 \Rightarrow \mu_+ > 1 \quad \text{this image is always magnified} \)

\( \mu_- < 0 \quad \text{mirror image, may be magnified or demagnified} \)

As \( y \rightarrow 0, \quad X_+ \rightarrow 1 (\phi_+ \rightarrow \phi_E) \) and \( \mu_+ \rightarrow \infty \)

\( X_- \rightarrow -1 (\phi_- \rightarrow -\phi_E) \) and \( \mu_- \rightarrow -\infty \)
Total magnification \( \mu_p = \mu_+ + |\mu_-| = \frac{y^2 + 2}{y\sqrt{y^2 + 4}} \)

For \( y = 1 \): \( \mu_+ = 1.171, \mu_- = -0.171 \Rightarrow \mu_p = 1.342 \)

The separation between images is \( \sqrt{y^2 + 4} \theta_E = 2\theta_E \), but in practice not much larger, so for \( y \gg 1 \), \( |\mu_-| \ll 1 \)
and this image cannot be seen.

Odd number theorem? The above applies also to an extended mass \( M \); but only for light rays that stay outside it. If the 1st ray passes through \( M \), it is deflected less and does not produce an image. If it passes outside \( M \), there will be a third ray, passing through \( M \), producing a third image. For a compact mass with \( y \gg 1 \),

\[ \mu = \frac{1}{(1-y)(1+y-2y)} \approx \frac{1}{(1-y)^2} \ll 1 \] for this third image.
The simplest model for the density profiles of galaxies and clusters produces flat rotation curves (exercise)

\[ \Sigma(r) = \frac{\sigma^2}{2\pi G r^2} \propto r^{-2} \]

\[ \Rightarrow \text{surface mass density} \quad (\text{exercise}) \]

\[ \Sigma(\infty) = \int_0^\infty 4\pi \sigma^2 \left( \frac{1}{\sqrt{r^2 + \sigma^2}} \right) \, dr = \frac{6\sigma^2}{2\pi G} \propto \Sigma^{-1} \]

\[ \mathcal{H}(\theta) = 2\pi \frac{D_{\odot}}{D_S} \frac{\sigma^2}{\Sigma} \frac{1}{10^2} \]

\[ \Rightarrow 1 + \mathcal{H} - 2\bar{n} = 1 \]

no radial critical curves

\[ \mathbf{\alpha}(\theta) \cdot \mathbf{\bar{\alpha}}(\theta) = \mathbf{\bar{\alpha}}(\theta) \cdot \mathbf{\bar{\alpha}}(\theta) = \frac{\sigma^2}{10^2} \]

has constant magnitude \( \mathcal{H} \)

\[ \tan \theta = \frac{\mathcal{H}}{10^2} \]

\[ \text{Lens equation} \quad \beta = \theta - \alpha = \theta - \mathcal{H} \cdot \frac{\sigma}{10^2} \]

\[ \text{or} \quad y = x - \frac{x}{1x1} \quad \text{where} \quad x = \frac{\sigma}{\mathcal{H}}, \quad y = \frac{\beta}{\mathcal{H}} \]

\[ \text{choose } y > 0 \]

two solutions \( x = y+1 \) for \( 0 < y < 1 \)

one solution \( x = y+1 \) for \( y > 1 \)

\[ \text{Magnification} \quad \mu_+ = \frac{1}{1 - \bar{n}}(1 + \bar{n} - 2\bar{n}) = \frac{1}{1 - \bar{n}} = \frac{1x1}{1x1 - 1} \]

\[ \mu_+ = \frac{y+1}{y} > 0 \]

\[ \mu_- = \frac{y-1}{-y} < 0 \]
Two strange features:

1) Odd-number theorem violated

2) #images changes by 1, when source crosses critical curve, not causal

Due to singularity \( g(r) \to \infty \) at \( r \to 0 \) \( \Rightarrow \alpha \) not continuous at \( \theta = 0 \)

\[ \beta = \theta - \alpha \]

Smooth the singularity into finite-density core \( \Rightarrow \alpha \) changes continuously

\( \Rightarrow \) a radial critical curve \( \frac{\partial \beta}{\partial \theta} = 0 \) and caustics appear at \( \beta_r < \theta_r \), \( \theta_r \) small

If the core region is small \( \Rightarrow \frac{\partial \alpha}{\partial \beta} \) and \( \frac{\beta}{\theta} \gg 1 \) \( \Rightarrow \) third image strongly demagnified

Parity of images determined by signs of \( \frac{\partial \beta}{\partial \alpha} \) and \( \frac{\beta}{\theta} \): both negative for third image \( \Rightarrow \mu > 0 \)
§2.9 Non-Symmetric Lenses

- Qualitative details of centrally condensed asymmetric lenses do not depend strongly on the radial profile.

- Breaking the symmetry leads to qualitatively new properties:

  central caustic point → finite curve, a source inside it may have 5 images.

Many observed lenses have 4 images; probably the 5th is invisible due to strong demagnification at the center.