The metric of a homogeneous and isotropic (FRW) universe can be written

\[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

\[ = -dt^2 + a(t)^2 \left[ dw^2 + f_K(w)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

where \( a(t) \) is the scale factor, \( K \) is the curvature constant, and

\[ f_K(w) \equiv \begin{cases} 
K^{1/2} \sinh(K^{1/2}w) & \text{for } K > 0 \quad (\text{closed universe}) \\
w & \text{for } K = 0 \quad (\text{flat universe}) \\
K^{-1/2} \sinh(1K^{1/2}w) & \text{for } K < 0 \quad (\text{open universe}) 
\end{cases} \]

We normalize the scale factor so that \( a_0 = a(t_0) = 1 \Rightarrow a(t) = \frac{1}{1+z} \)

Two alternative radial coordinates \( r \) and \( w \): \( r = f_K(w) \)

\( a(t)dr \) : transverse distance \( ds \) at time \( t \)

\( a(t)dw \) : radial distance \( ds \) at time \( t \)

\( rd\theta \) : transverse comoving distance

\( dw \) : radial comoving distance

Conformal time \( \eta \): \( d\eta \equiv adt \)

\[ ds^2 = a(\eta)^2 \left[ -d\eta^2 + \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

\[ = a(\eta)^2 \left[ -d\eta^2 + dw^2 + f_K(w)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

\( \Rightarrow \) radial light ray has \( dw = \pm d\eta = \pm \frac{dl}{a(t)} \)

post light cone \( w(\eta_0 - \eta) = w(\eta) = w(\xi) = \eta_0 - \eta \)
Hubble parameter $H(t)$ and Hubble constant $H_0 \equiv H(t_0)$

$$H \equiv \frac{\dot{a}}{a} = \frac{1}{a} \frac{da}{dt} = \frac{1}{a} \frac{d(1+z)}{dt} = -\frac{1}{1+z} \frac{dz}{dt} = -a \frac{dz}{dt}$$

$$H^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2} = H_0^2 \left[ \Omega_r \rho_c^4 + \Omega_m \rho_m^3 + (1-\Omega_r) \rho_c^2 + \Omega_d \right]$$

$$= H_0^2 \left[ \Omega_r (1+z)^4 + \Omega_m (1+z)^3 + (1-\Omega_r)(1+z)^2 + \Omega_d \right]$$

$$\Omega_m - 1 = KH_0^2$$

$$\frac{dH}{dt} = \frac{da}{a} = -\frac{dz}{1+z} \Rightarrow \frac{dz}{H} = -(1+z) \frac{dt}{a} = \frac{dt}{a} = \frac{da}{a^2 H}$$

**Comoving distances between two objects at redshifts $z_1$ and $z_2$ along the same line of sight:**

$$D^c(z_1, z_2) = \int w(z) \frac{dz}{H(z)} = \int \frac{dz}{a(z)H(z)} = \int \frac{da}{a^2 H(a)} = w(z_2) - w(z_1)$$

**Comoving distances along the same line of sight are additive:**

$$D^c(z_2) = D^c(0, z_2) = D^c(z_1) + D^c(z_1, z_2)$$
§3.2 Distances in Cosmology

Here we refer to distances between objects at different redshifts along the same line of sight.

**Angular diameter distance** $D^A$

$$D^A(z) \text{ from here to } z = \frac{\Delta s}{\Delta \theta} = a(t) r(z) = \frac{r(z)}{1+z} = \frac{f_K(w(z))}{1+z} = \frac{1}{1+z} f_K(D^C(z))$$

$D^A(z_1, z_2)$ from $z_1$ to $z_2$: move origin to object seen at $z_1$

$$\Rightarrow w \text{ at } z_2 \text{ is } D^C(z_1, z_2) \Rightarrow D^A(z_1, z_2) = \frac{1}{1+z_2} f_K(D^C(z_1, z_2))$$

**Comoving angular diameter distance** $D^{CA}$

$$D^{CA}(z) = r(z) = f_K(w(z)) = f_K(D^C(z)) = (1+z) D^A(z)$$

$$D^{CA}(z_1, z_2) = f_K(D^C(z_1, z_2)) = (1+z_2) D^A(z_1, z_2)$$

**Luminosity distance** $D^L$

$$D^L(z) = \sqrt{\frac{L}{4\pi}} \quad \text{(L absolute luminosity, } L \text{ apparent luminosity)}$$

$$D^L(z) = (1+z) D^{CA}(z) = (1+z)^2 D^A(z)$$

Surfina brightness $\propto \frac{\text{flux density}}{\text{solid angle}} \propto \frac{D^A(z)^2}{D^L(z)^2} = (1+z)^{-4}$ decreases fast with increasing redshift.

If $K \neq 0$, $D^A, D^{CA}, D^L$ are not additive: $\sin(x+y) \neq \sin x + \sin y$

If $K = 0$ (flat universe) then $D^{CA} = D^C$ and two additive:

$$\Rightarrow D^{CA}(z_2) = D^{CA}(z_1) + D^{CA}(z_1, z_2)$$

$$D^A(z_2) = \frac{1+z_1}{1+z_2} D^A(z_1) + D^A(z_1, z_2)$$

$$D^L(z_2) = \frac{(1+z_1)^2}{1+z_2} D^L(z_1) + D^L(z_1, z_2)$$

$$\Rightarrow w = r \Rightarrow ds^2 = a(y)^2 [-dx^2 + dw^2 + dy^2 + sin^2(dy)^2] = a(y)^2 [-dy^2 + dw^2 + dy^2 + dz^2]$$

Light rays travel in these cones exactly as in Minkowskian space (as function of $y$ instead of $t$).
3.3 Cosmological Distances and Lensing

The distances $D_{a}, D_{as}, D_{s}$ should be interpreted as $D^{A}$

$$\hat{\alpha} = \frac{S}{D^{A}(z_{a}, z_{s})} \Rightarrow \hat{\alpha} = \frac{S}{D^{A}(z_{s})} = \frac{D^{A}(z_{a}, z_{s})}{D^{A}(z_{s})}$$

$$\Sigma_{cr} = \frac{1}{4\pi G} \frac{D^{A}(z_{s})}{D^{A}(z_{a}) D^{A}(z_{a}, z_{s})} = \frac{1+z_{a}}{4\pi G} \frac{D^{CA}(z_{s})}{D^{CA}(z_{a}) D^{CA}(z_{a}, z_{s})}$$

$$\Theta_{E} = \frac{\sqrt{4GM D^{A}(z_{a}, z_{s})}}{D^{A}(z_{a}) D^{A}(z_{s})} = \sqrt{\frac{4GM(1+z_{a})}{D^{CA}(z_{a}) D^{CA}(z_{a}, z_{s})}}$$

Observationally, we have so far not been able to distinguish our universe from the $K=0$ case.

$\Rightarrow$ $K$ must be small $\Rightarrow$ the difference between $D^{CA}$ and $D^{C}$ must be small.

$\Rightarrow$ We can approximate $D^{CA} \approx D^{C}$, unless we want to do accurate work, e.g., trying to measure $K$.

$|\Omega_{-1}| < 0.005 \ (\text{Planck+BAO 2015}) \Rightarrow |K| < 0.005 H_0^2$

$R_{\text{min}} = \frac{1}{\sqrt{|K|}} > 1.4 H_0$
§3.4 Time Delay

- The light signal from the source to the observer is delayed by the gravitational lens, compared to absence of the lens. The delay can be divided into two parts:

1) "geometric" delay: due to different length of light path in the background FRW geometry
2) "potential" delay: occurs at the lens and is due to local perturbation of the metric by the lens. This is also a geometrical effect, but in perturbation theory the metric perturbation is mapped into potentials.

**Geometric delay in the flat \( (K=0) \) universe**

The light paths with and without lens:

\[
D_{\alpha}^2 + (D_{\alpha}^S)^2 = \sqrt{D_{\alpha}^2 + (D_{\alpha}^S)^2}
\]

\[
D_{\alpha}^C = D_{\alpha}^S
\]

\[
\beta D_{\alpha}^C = \beta D_{\alpha}^S
\]

The time delay \( \Delta \eta_{geo} \) is given by the difference in the path lengths (in \( c=1 \) units) using comoving distances and conformal time:

\[
\Delta \eta_{geo} = \sqrt{D_{\alpha}^2 + (D_{\alpha}^S)^2} - \sqrt{D_{\alpha}^C + (\beta D_{\alpha}^C)^2}
\]

\[
\Delta \eta_{geo} = \ldots \approx \frac{1}{2} \frac{D_{\alpha}^S D_{\alpha}^C}{D_{\alpha}^S} (\beta - \beta)^2 \quad \text{(exercise; use small-angle approximation)}
\]

Within \( K=0 \) one needs spherical/hyperbolic trigonometry. The result is the same, except has

\[
D_{\alpha}^C \text{ instead of } D^C:
\]

\[
\Delta \eta_{geo} = \frac{1}{2} \frac{D_{\alpha}^C D_{\alpha}^S}{D_{\alpha}^C} (\beta - \beta)^2
\]
Potential delay in the flat ($K = 0$) universe

- Happens locally at the lens. The geometry of the lens perturbs the spacetime around locally.

In relativistic (cosmological) perturbation theory, we find that the metric can be written as

$$ds^2 = c^2(y)^2 \left[-(1+2\Phi)dy^2 + (1-2\Psi)(dx^2 + dy_\perp^2 + dz^2)\right]$$

$$r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where the Borel potentials $\Phi(y, \vec{r})$ and $\Psi(y, \vec{r})$ are small, $|\Phi| \ll 1, |\Psi| \ll 1$ in the Newtonian limit $\Phi = \Psi$ and is equal to the Newtonian gravitational potential.

In the Newtonian limit, we can assume this setup constant in time while the light ray passes.

For the light ray, $ds^2 = 0 \Rightarrow (1+2\Phi)dy^2 = (1-2\Psi)dx^2$

$$\Rightarrow dy = \sqrt{\frac{1-2\Psi}{1+2\Phi}} dx \Rightarrow (1-2\Psi) dx = (1-2\Psi)(1+z_d) dr_3$$

- Comoving distance

For an overdensity, $\Phi < 0$ \Rightarrow the perturbation causes a time delay

$$\Delta t_{\text{pot}} = -2(1+z_d) \int \phi dr_3$$

We use this to compare different light rays passing through the same lens. The integral should also be such that it's far enough from the lens so that the disturbance to $\phi$ becomes negligible.

(But not $\int_{-\infty}^{\infty}$; we assume that $\phi = 0$ (no perturbation) on average far from the lens.)

\footnote{Here $r$ is physical distance, not comoving distance}

\footnote{Scalar perturbations, Newtonian gauge}
Define the lensing potential \( \psi(\vec{\theta}) = \frac{D^A_{\text{ds}}}{D^A_{\text{ds}}} \cdot 2 \int \psi(D^A_{\text{ds}}, r_3) \, dr_3 \)

\[
\Delta \eta_{\text{pot}} = -(1+z_1) \frac{D^A_{\text{ds}} D^A_{\text{ds}}}{D^A_{\text{ds}}} \psi(\vec{\theta}) = -\frac{D^C_{\text{ds}} D^C_{\text{ds}}}{D^C_{\text{ds}}} \psi(\vec{\theta})
\]

We derived this for \( k=0 \); but the result holds also for \( k \neq 0 \).

This lensing potential \( \psi(\vec{\theta}) \) is the same as the deflection potential \( \psi(\vec{\theta}) \) defined earlier (we skip the proof).

\[
\Delta \eta = \Delta \eta_{\text{geo}} + \Delta \eta_{\text{pot}} = \frac{D^C_{\text{ds}} D^C_{\text{ds}}}{D^C_{\text{ds}}} \left[ \frac{1}{2} (\vec{v}_1 - \vec{v}_2)^2 - \psi(\vec{\theta}) \right]
\]

\[
= \frac{D^C_{\text{ds}} D^C_{\text{ds}}}{D^C_{\text{ds}}} \tau(\vec{\theta}_1; \vec{\theta}_2) = \Delta t \quad \text{(at the observer, since for her)} \quad \Delta t = c_0 \Delta \eta = \Delta \eta
\]

\( \tau \) is the Friedmann potential.

What can be observed (if some observable event happens at the same), is the time delay difference between two images:

\[
\Delta t(\vec{\theta}_1; \vec{\theta}_2) = \frac{D^C_{\text{ds}} D^C_{\text{ds}}}{D^C_{\text{ds}}} \left[ \tau(\vec{\theta}_1; \vec{\theta}_2) - \tau(\vec{\theta}_2; \vec{\theta}_2) \right]
\]

If we understand the lens structure well enough to estimate \( \tau(\vec{\theta}_1; \vec{\theta}_2) - \tau(\vec{\theta}_2; \vec{\theta}_2) \), the time delay can be used to determine cosmological parameters, especially \( H_0 \), on which \( D^C(z_1), D^C(z_2), D^C(z_3; z_5) \) depend.