1. a) Let $A$ be antisymmetric real $n \times n$ matrix and $v$ $n$-dimensional vector ($v \in \mathbb{R}^n$). Show that the vector $u = Av$ is orthogonal against vector $v$, or $v^T u = 0$.

$A$ antisymmetric: $A^T = -A$. Now $v^T u = v^T Av$. On the other hand,

$v^T u = u^T v = (Av)^T v = v^T A^T v = -v^T Av$.

Thus, we must have $v^T u = 0$.

b) Let $B$ be Hermitean $n \times n$ matrix, which also satisfies the condition $B^2 = B$. Show that the vectors $Bv$ and $(I - B)v$ are orthogonal for any vector $v \neq 0$ ($v \in \mathbb{C}^n$).

Recall that complex vectors are orthogonal if $v^\dagger u = 0$.

Now $(Bv)^\dagger (1 - B)v = v^\dagger B^\dagger (1 - B)v = v^\dagger B(1 - B)v = v^\dagger (B - B)v = 0$.

Thus, vectors are orthogonal.

2. Find the inverses of matrices

\[
\begin{align*}
a) & \quad A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \\ b) & \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]

a) Matrix is diagonal, thus the inverse has just inverse diagonals: $A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$.

b) Use the Gauss elimination:

\[
\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}
\begin{array}{c}
\text{subtract row 2} \\
\text{subtract row 3}
\end{array}
\Rightarrow\begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}
\]

Thus, $B^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

3. Find the eigenvalues and normalized eigenvectors of the matrix

\[
M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\]

Answer: eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$; and corresponding normalized eigenvectors $v_1 = \frac{1}{\sqrt{2}}(1, -1)^T$, $v_2 = \frac{1}{\sqrt{2}}(1, 1)^T$.

Eigenvalues are obtained from
\[
\begin{vmatrix}
2 - \lambda & 1 \\
1 & 2 - \lambda
\end{vmatrix} = (2 - \lambda)^2 - 1 = 0 \Rightarrow \lambda_1 = 1, \ \lambda_2 = 3
\]

Eigenvectors, eg. for \(\lambda_1 = 1\):
\[
\begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix} = \lambda_1 \begin{pmatrix}
a \\
b
\end{pmatrix} \Rightarrow 2a + b = a \Rightarrow b = -a,
\]
and the normalized eigenvector can be chosen as \(v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\).

4. Matrix diagonalization:

Take the normalized eigenvectors \(v_1\) ja \(v_2\) from the previous question (column vectors) and construct a matrix \(P = \langle v_1, v_2 \rangle\). Form its inverse \(P^{-1}\) (hint: check the matrix \(P^T\)). Calculate now the matrix product \(A = P^{-1}MP\).

If you have done everything right, \(A\) is a diagonal matrix with the eigenvalues of \(M\) along the diagonal. This process is called the diagonalization of the matrix \(M\) using similarity transformations.

If the matrix \(M\) is symmetric (as above), the matrix \(P\) is orthogonal, i.e. \(P^{-1} = P^T\). If \(M\) is Hermitean (\(M^\dagger = M\)), then \(P\) is unitary, \(P^{-1} = P^\dagger\).

Now \(P^T = P^{-1}\), which can be seen by explicit computation:
\[
P^T = \frac{1}{2} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix} = I
\]

However, in general if \(P = \langle v_1, v_2 \rangle\), where \(v_i\) are orthogonal eigenvectors with eigenvalues \(\lambda_i\), we also have
\[
P^T = \begin{pmatrix}
v^T_1 \\
v^T_2
\end{pmatrix} \begin{pmatrix}
\lambda_1 v_1 \\
\lambda_2 v_2
\end{pmatrix} = \begin{pmatrix}
\lambda_1 v^T_1 v_1 \\
\lambda_2 v^T_2 v_2
\end{pmatrix} = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}
\]

Thus, we get a diagonal matrix with eigenvalues on the diagonal. In this case, \(A = \begin{pmatrix}
1 & 0 \\
0 & 3
\end{pmatrix}\).

5. Matrix functions for square matrices can be defined using the Taylor series of the functions, e.g.
\[
\exp M = \sum_{n=0}^{\infty} \frac{M^n}{n!}
\]

Let us now assume that we can diagonalize the matrix \(M\) as in the previous question, i.e. we can write \(M = PDP^{-1}\), where \(D\) is a diagonal matrix whose elements are eigenvalues of \(M\).
a) Show that $M^n = PD^nP^{-1}$.

Now $M^2 = MM = PD P^{-1}PD P^{-1} = PD^2 P^{-1}$

Clearly, we can repeat this as many times as we wish, and thus $M^n = PD^nP^{-1}$.

What kind of matrix is $D^n$?

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, and it is immediately clear that

$D^2 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$.

Thus, repeating this $D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$.

b) Then show that $e^M = Pe^DP^{-1}$. What is $e^D$?

$e^M = \sum_n \frac{M^n}{n!} = \sum_n \frac{PD^n P^{-1}}{n!} = P \sum_n \frac{D^n}{n!} P^{-1} = Pe^DP^{-1}$.

Further, $e^D = \begin{pmatrix} \sum_n \frac{\lambda_1^n}{n!} \\ 0 \\ \sum_n \frac{\lambda_2^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$

c) Apply this to matrix $M$ of question 3.

$$e^M = Pe^DP^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^1 & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} e^1 & e^3 \\ -e^1 & e^3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^1 + e^3 & -e^1 + e^3 \\ -e^1 + e^3 & e^1 + e^3 \end{pmatrix}$$

6. Find the eigenvalues and -vectors for the 2-dimensional rotation matrix

$$M = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$$

Answer: eigenvalues are $\lambda_{1,2} = e^{\pm i\phi}$, and normalized eigenvectors $v_{1,2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$. 